16% for Problem 1-5 and 20% for Problem 6.

1. Let $x^2 + y^2 = r^2,\ r > 0$. Evaluate
$$\left|\frac{y''}{[1+(y')^2]^{3/2}}\right|.$$

2. Find maximum area of the rectangle that can be inscribed in the portion of parabola $y^2 = 4px$ intercepted by the line $x = a$ where $p, a > 0$.

3. Suppose that $C(x)$ and $S(x)$ are two functions such that $C'(x) = -S(x)$ and $S'(x) = C(x)$. Prove that
$$C^2(x) + S^2(x)$$
is constant.

4. Find
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\arcsin(i/n)}{n}.$$

5. Let $r$ is a real number not equal to an integer. Find
$$\sum_{k=1}^{\infty} \frac{\binom{r}{k}}{2^k}.$$

6. Find the volume of
$$S = \{(x_1, x_2, x_3, x_4) \,|\, x_1^2 + x_2^2 + x_3^2 + x_4^2 \le 1\}.$$

科目：線性代數【應數系二年級】



第1~4題各15分，其餘每題10分，無計算或證明過程者，不予計分。

1. Find all eigenvalues of $A=\begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$. What is the necessary and sufficient condition for $A$ to be symmetric positive definite?

2. Let $A$ be an $m\times n$ matrix with rank $r$. Find the necessary and sufficient conditions on $m, n, r$ such that the number of solutions to $A\mathbf{x}=\mathbf{b}$ is ① 0 or 1 depending on $\mathbf{b}$, ② infinite for every $\mathbf{b}$, ③ 0 or infinite depending on $\mathbf{b}$, ④ 1 for every $\mathbf{b}$.

3. Find the projection matrix onto $x+y+z=0$ in $\mathbb{R}^3$. What is the Jordan form of this matrix?

4. Find the conditions on $a, b, c, d$ such that $\begin{cases} x+y+2z=b \\ 2x+ay-3z=c \\ 3x+6y-5z=d \end{cases}$ has ① unique solution, ② infinitely many solutions, ③ no solution.

5. Let $F(x,y)=(x+y,\ x-y)$, find its matrix representation with respect to the new basis obtained by rotating the standard basis $60^\circ$ counter-clockwise about the origin.

6. Consider the standard complex inner product in $\mathbb{C}^4$. Transform $\left\{\begin{pmatrix}1\\0\\i\\0\end{pmatrix}, \begin{pmatrix}i\\0\\2\\0\end{pmatrix}, \begin{pmatrix}1\\-i\\0\\-1\end{pmatrix}\right\}$ into an orthonormal set.

7. Find the least squares straight line fit to data $(0,0)$, $(1,2)$, $(2,7)$, $(3,10)$.

8. Assume $2\times 2$ matrix $A$ has eigenvalues $\pm\sqrt{2}\,i$, find $A^n$ for all integer $n$.