

(一) 填充題：每個空格4分，共計60分。此部份祇需將答案寫在答案卷上，並在答案前標明每個空格的英文字母代號；不需列出計算過程

1. Evaluate the following limits :

(i)  $\lim_{x \rightarrow 0} \frac{2 \cos x - 2 + x^2}{x^2 - \ln(1 + x^2)} = \underline{\hspace{2cm}} \text{ (A)}$       (ii)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{4n^2 - k^2}} = \underline{\hspace{2cm}} \text{ (B)}$

2. Evaluate the following integrals :

(i)  $\int_1^2 x\sqrt{2-x} dx = \underline{\hspace{2cm}} \text{ (C)}$       (ii)  $\int_0^1 \left\{ \int_{\sqrt{x}}^1 \frac{dy}{1+y^3} \right\} dx = \underline{\hspace{2cm}} \text{ (D)}$

3. Let  $p$  and  $q$  be two real numbers, and let  $f(x) = e^{x^2-1} + px + q$  for all  $x \in \mathbb{R}$ . If  $f(1) = 4$  is a local extremum of  $f$ , then  $p = \underline{\hspace{2cm}} \text{ (E)}$ ,  $q = \underline{\hspace{2cm}} \text{ (F)}$

4. The length of the curve  $y = \ln \cos x$  ( $0 \leq x \leq \frac{\pi}{4}$ ), is  $\underline{\hspace{2cm}} \text{ (G)}$

5. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function. Assume that  $f$  is differentiable at the point  $(1, 2) \in \mathbb{R}^2$  with  $f(1, 2) = 0$ ,  $f_x(1, 2) = -1$  and  $f_y(1, 2) = 3$ .

(i) The equation of the tangent plane to the surface  $z = f(x, y)$  at the point  $(1, 2, 0)$  is  $\underline{\hspace{2cm}} \text{ (H)}$

(ii) If  $\vec{u} = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ , then the directional derivative of  $f$  at  $(1, 2)$  in the direction  $\vec{u}$  is  $\underline{\hspace{2cm}} \text{ (I)}$

(iii) If  $g(t) = f(1 - 4 \tan^{-1} t, 3 - e^{2t})$  for  $t \in \mathbb{R}$ , then  $g'(0) = \underline{\hspace{2cm}} \text{ (J)}$

6. If  $\Omega = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 2, 0 \leq y \leq 1\}$ , then

$$\iint_{\Omega} x e^{xy} dx dy = \underline{\hspace{2cm}} \text{ (K)}$$

7. If

$$f(x) = 3 + \int_2^{\sqrt{x}} \frac{dt}{1 + 3t^2 + t^4} \quad \text{for } x > 0,$$

and if  $g$  is the inverse of  $f$ , then  $f'(x) = \underline{\hspace{2cm}} \text{ (L)}$ , and  $g'(3) = \underline{\hspace{2cm}} \text{ (M)}$

8. The interval of convergence of the power series  $\sum_{k=3}^{\infty} \frac{\ln k}{k} (x-2)^k$  is  $\underline{\hspace{2cm}} \text{ (N)}$

9. If  $f(x) = \cos(x^2)$ , then  $f^{(12)}(0) = \underline{\hspace{2cm}} \text{ (O)}$

(二) 計算題：請詳列計算過程，否則不計分

I. Let

$$f(x) = \begin{cases} \frac{1 - \cos x}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that  $f$  is differentiable on  $\mathbb{R}$ , and find  $f'(x)$ . (10%)

II. Evaluate the value of the integral :

$$\int_0^{\infty} \frac{dx}{(x^2 + 1)(x^2 + 4)} \quad (10\%)$$

III. Let  $\Omega = \{(x, y) : x^2 + y^2 \leq \frac{1}{4} \text{ and } x \geq 0\}$ . Evaluate the double integral :

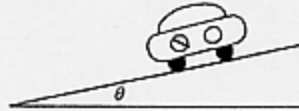
$$\iint_{\Omega} \sin^{-1}(x^2 + y^2) dx dy \quad (10\%)$$

IV. (i) If  $g(t) = t^2 - \frac{t^4}{3} - \sin^2 t$ , prove that  $g(t) \leq 0$  for all  $t \in \mathbb{R}$ . (5%)

(ii) Use (i) to evaluate the limit :  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin^2 x + \sin^2 y}{x^2 + y^2}$  (5%)

1. A balloon is rising at 10 m/s when its passenger throws up a ball at 12 m/s. How much later does the passenger catch the ball? (10%)

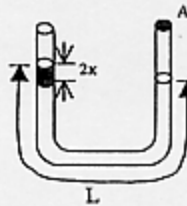
2. A car travels in a horizontal circle around a curve of radius  $r$  banked at an angle  $\theta$  to the horizontal. If  $\mu$  is the coefficient of static friction, find the maximum speed without sliding. (10%)



3. Find the center of mass of the flat semicircular plate of radius  $R$ . (15%)

4. The water is slightly displaced and then allowed to move freely. (15%)

- (a) Show that the liquid executes simple harmonic motion.
- (b) What is the period?



5. A charged disk of radius  $R$  with an uniform charge density  $\sigma$  over its surface. (20%)

- (a) What is the potential at a point  $P$  on the disk axis, a distance  $z$  from the disk?
- (b) What is the electric field at a point  $P$  derived from the electric potential directly?

6. Find the magnetic field at an arbitrary point  $P$  on the axis of a circular loop of radius  $a$  carrying current  $I$ . (15%)



7. An oscillating  $LC$  circuit consisting of a  $1.0 \text{ nF}$  capacitor and a  $3.0 \text{ mH}$  coil has a maximum voltage of  $3.0 \text{ V}$ . (15%)

- (a) What is the maximum charge on the capacitor?
- (b) What is the maximum current through the circuit?
- (c) What is the maximum energy stored in the magnetic field of the coil?