科目：基礎數學【應數系碩士班甲組】 √

共七題。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. (10%) Let $T:\mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by
$$T(x,y,z,w)=(x+y,z+w,0).$$
Find the image and kernel of $T$.

2. (15%) Let $(\mathbb{R}^4, <\,,\,>)$ be an inner product space with $<u,v>= u_1v_1+u_2v_2+u_3v_3+u_4v_4$, $u=(u_1,u_2,u_3,u_4)$ and $v=(v_1,v_2,v_3,v_4)$. Let
$$V=\{(x,y,z,w)\in\mathbb{R}^4 \mid x+y+z+w=0\}$$
be a subspace of $\mathbb{R}^4$. Find an orthonormal basis for $V$.

3. (15%) Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be the linear transformation given by
$$T(x,y,z)=(3x-2y-2z,2x-y-2z,2x-2y-z).$$
Find a basis such that the matrix representation of $T$ is a diagonal form with respect to the found basis.

4. (15%) Let $f(x)=\begin{cases} ax^2, & x<1 \\ 2x-b, & x\geq 1. \end{cases}$ Find $a$ and $b$ such that $f$ is differentiable on $\mathbb{R}$.

5. (15%) Evaluate
$$\int_{\ln 4}^{\infty}\frac{dx}{e^x-5+6e^{-x}}.$$

6. (15%) Evaluate
$$\int_0^2\int_{y^2}^4 \sqrt{x}\sin\pi x\; dxdy.$$

7. (15%) Find the interval of convergence of $\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{\ln n}$, $\quad x\in\mathbb{R}$.

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Master Entrance Exam- Mathematical Statistics, 2009

(1) Let $X_i, i = 1,2,3$ be three independent Gamma$(\alpha, \beta)$ random variables with density function $f(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$. Let

$$Y_1 = \frac{X_1}{X_1+X_2}, Y_2 = \frac{X_1+X_2}{X_1+X_2+X_3}, Y_3 = X_1+X_2+X_3.$$

Show that $Y_i, i = 1,2,3$ are independent and find their marginal distributions. (10 pts)

(2) Let $X_1, \cdots, X_n$ be an i.i.d. sample from a Rayleigh distribution with density function, $f(x) = \frac{x}{\vartheta^2}e^{-x^2/2\vartheta^2}, x \geq 0$.

(a) Find the method of moment estimate of $\vartheta$.(10 pts)

(b) Find the maximum likelihood estimate of $\vartheta$.(10 pts)

(c) Find the Cramer-Rao lower bound on the variance of any unbiased estimate of $\vartheta$.(10 pts)

(3) Assume $X_1, \cdots, X_n$ are independent $N(\mu, \sigma^2)$ random variables. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and $\vec{X} = (X_1 - \bar{X}, X_2 - \bar{X}, \cdots, X_n - \bar{X})$.

(a) Find the following joint moment generating function of $\bar{X}$ and $\vec{X} = (X_1 - \bar{X}, X_2 - \bar{X}, \cdots, X_n - \bar{X})$,

$$M(s, t_1, \cdots, t_n) = E[\exp(s\bar{X} + t_1(X_1 - \bar{X}) + \cdots + t_n(X_1 - \bar{X}))].(10 \text{ pts})$$

(b) Use the result of (a) to find the distribution of the sample mean $\bar{X}$ and show that the sample mean $\bar{X}$ and the sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$ are independent.(10 pts)

(4) Let $X_1, \cdots, X_n$ be a iid sample from a distribution $F(x)$, and let $F_n(x) = \frac{1}{n}\sum_{i=1}^n I_{(-\infty,x]}(X_i)$, where

$$I_{(-\infty,x]}(X_i) = \begin{cases} 1 & \text{if } X_i \leq x \\ 0 & \text{if } X_i > x \end{cases}$$

Find the correlation of $F_n(u)$ and $F_n(v)$.(10 pts)

(5) Suppose that $X_1, X_2, \cdots, X_n$ are iid $N(\mu, \sigma^2)$ random variables.

(a) Show that the statistic $T = (\bar{X}, S^2)$ (the sample mean and sample variance) is a complete sufficient statistic for $(\mu, \sigma^2)$.(10 pts)

(b) Derive the generalized likelihood ratio test for $H_0 : \sigma^2 = \sigma_0^2$ versus $H_a : \sigma^2 \neq \sigma_0^2$.(10 pts)

(6) Let $X_1, \cdots, X_n$ be a sample from $U(a, b)$ distribution and denote the order statistics by $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$. Find the mean and variance of the $kth$ order statistic $X_{(k)}$.(10 pts)

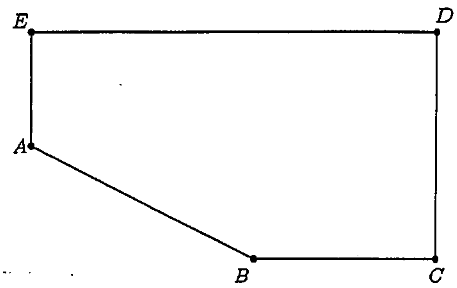
科目：機率論【應數系碩士班甲組】✓

共十題，每題10分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. The Gnollish language consists of 3 words, "splargh," "glumph," and "amr." In a sentence, "splargh" cannot come directly before "glumph"; all other sentences are grammatically correct (including sentences with repeated words). How many valid 3-word sentences are there in Gnollish?

2. Six points on a circle are given. Four of the chords joining pairs of the six points are selected at random. What is the probability that the four chords are the sides of a convex quadrilateral? (Simplify your answer)

3. A point $P$ is selected at random from the interior of the pentagon with vertices $A=(0,2), B=(4,0), C=(2\pi+1,0), D=(2\pi+1,4)$ and $E=(0,4)$. What is the probability that $\angle APB$ is obtuse?




4. English and American spellings are *rigour* and *rigor*, respectively. A man staying at a Parisian hotel writes this word, and a letter taken at random from his spelling is found to be a vowel. If 40 percent of the English-speaking men at the hotel are English and 60 percent are Americans, what is the probability that the writer is an Englishman?

5. On a multiple-choice exam with 3 possible answers for each of the 5 questions, what is the probability that a student would get 4 or more correct answers just by guessing? (Simplify your answer)

6. The lifetime of an electronic amplifier is modeled as an exponential random variable. If 10% of the amplifiers have a mean of 20,000 hours and the remaining amplifiers have a mean of 50,000 hours, what proportion of the amplifiers fail before 60,000 hours?

7. Suppose $X_1, X_2, \ldots$ are jointly continuous and independent, each distributed with marginal pdf $f(x)$, where each $X_i$ represents annual rainfall at a given location. Find the distribution of the number of years until the first year's rainfall, $X_1$, is exceeded for the first time.

8. Find the pdf of $\prod_{i=1}^{n} X_i$, where the $X_i$s are independent uniform$(0,1)$ random variables.

9. If $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables having uniform distributions over $(0,1)$, find (a) $E[\max_{i=1,\ldots,n} X_i]$; (b) $E[\min_{i=1,\ldots,n} X_i]$.

10. Let $X$ be a nonnegative random variable. Prove that

$$E[X] \le (E[X^2])^{1/2} \le (E[X^3])^{1/3} \le \cdots$$

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科目：線性代數【應數系碩士班乙組、丙組】 √

Let $M_n(\mathbb{C})$ denote the set of $n \times n$ matrices over $\mathbb{C}$.

1. (40 points) Let $A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \in M_{12}(\mathbb{C})$.

   (a) Find the characteristic polynomial of $A$.

   (b) Find the minimal polynomial of $A$.

   (c) Is $A$ diagonalizable ? (Give your reasons).

   (d) Find the Jordan form of $A$.

2. (50 points) Let $X \in M_{13}(\mathbb{C})$. We denote the rank of $X$ by rank$X$ and denote the transpose of $X$ by $X^t$. Prove or disprove the followings:

   (a) rank$(X^2)$=rank$X$.

   (b) rank$(X^tX)$=rank$X$.

   (c) if 0 is an eigenvalue of $A$, then $A$ is singular.

   (d) if $A$ is singular, then 0 is an eigenvalue of $A$.

   (e) if there exists a positive integer $m$ such that $A^m = 0$, then $A^{13} = 0$.

3. (10 points) Find the characteristic polynomial of $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}^{100}$.

科目：微積分【應數系碩士班乙組】√

**Entrance Calculus for the Master program**

Twenty points for each problem. Please write down all the detail of your proof and answers.

1. Find the limit of the following sequence $\{x_N\}$,

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, ..., \sqrt{2+\sqrt{2+\sqrt{2+...}}} \text{ involving } n \text{ sqare roots }, ...$$

2. Determine the convergence or the divergence of the following series,

$$\ln \Pi_{n=1}^{\infty}(1+\frac{1}{n}).$$

3. Prove the Schwarz inequality

$$\{\int_a^b wf(x)g(x)dx\}^2 \leq \{\int_a^b wf^2(x)dx\}\{\int_a^b wg^2(x)dx\}, \tag{1}$$

where $w \geq 0$ on $[a,b]$.

4. Find the sum of the limit

$$\lim_{n\to\infty}(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+...+\frac{n}{n^2+n^2}).$$

5. Prove that the functions $r^n \cos n\theta$ and $r^n \sin n\theta$ satisfy the Laplace equation,

$$\Delta u = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0,$$

where $(r,\theta)$ are the polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$.

科目：數值分析【應數系碩士班乙組】✓

*Please write down all the detail of your computation and answers.*

1. (20%) Use (1) Lagrange formula, (2) Neville's method, and (3) Newton divided difference formula to compute the cubic polynomial $p(x)$ interpolating the following data

| $x$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|
| $y$ | 2 | 2 | 2 | 8 |

2. (20%) (1) State the secant method, Newton method and the fixed point method to find a root of a given nonlinear equation. (2) State the advantage, disadvantage and the order of convergence of each method.

3. (20%) State (1) composite midpoint rule, (2) composite trapezoidal rule, (3) composite Simpson's rule and (4) two-point Gaussian quadrature to approximate an definite integral with error formulas.

4. (20%) (1) State the pivoting strategies of Gaussian elimination with maximal column pivoting (partial pivoting), scaled column pivoting (implicit scaling) and maximal pivoting (complete pivoting) for $n \times n$ matrix $A$.

   (2) Which one has the highest accuracy to solve linear systems?

   (3) Which one uses the least CPU time?

5. (20%) Let $A$ be an $n \times n$ nonsingular matrix and $B$ be an $n \times m$ matrix. State the fast numerical method to compute (1) determinant of $A$, (2) $A^{-1}B$. How many arithmetic operations are needed in (1), and what is the minimal memory needed in (2)?

科目：高等微積分【應數系碩士班丙組】 √

Solve all the problems with details. Each problem carries 20 points

1. A function $f$ is said to be Lipschitz on an interval $[a, b]$ if there is some $K > 0$ such that for all $x, y \in [a, b]$,

$$|f(x) - f(y)| \leq K|x - y|.$$

   (a) Show directly that the Lipschitz function $f$ is integrable on $[a, b]$. (10%)

   (b) Show that the Lipschitz function $f$ is uniformly continuous on $[a, b]$. (10%)

2. Given a sequence $(x_n)$ in $\mathbf{R}$, define

$$\overline{\lim}\, x_n := \lim_{n\to\infty} \sup\{x_k : \ k \geq n\} \qquad \text{and} \qquad \underline{\lim}\, x_n := \lim_{n\to\infty} \inf\{x_k : \ k \geq n\} .$$

   (a) Find $\overline{\lim}\, x_n$ and $\underline{\lim}\, x_n$ of the sequence $x_n = \sin(\frac{n\pi}{4})$. (6%)

   (b) For any bounded sequence $(x_n)$, show that $x = \overline{\lim}\, x_n$ if and only if for any $\epsilon > 0$, there is only finitely many $x_n$'s satisfying $x_n \geq x + \epsilon$ but there is infinitely many $x_n$'s satisfying $x_n \geq x - \epsilon$. (14%)

3. Show that if $f : \ I \to \mathbf{R}$ is continuous on $I$ where $I$ is a closed and bounded subset of $\mathbf{R}^n$, then $f[I]$ is also a closed and bounded set in $\mathbf{R}$.

4. Define the function $f : \ \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x, y) = \begin{cases} \frac{y\sqrt{x^2+y^2}}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   (a) Show that $f$ is not continuous at $(0, 0)$. (6%)

   (b) Show that $f$ has directional derivatives in all directions at $(0, 0)$. (7%)

   (c) Show that for all $c \in \mathbf{R}$, there is a vector $\mathbf{v}$ of norm 1 such that the directional derivative along $\mathbf{v}$, $D_{\mathbf{v}} f(0, 0)$ is equal to $c$. (7%)

5. (a) Evaluate the line integral $\int_C y\,dx + x\,dy$ when (10%)

      (i) $C$ is the upper semi-circle of radius 2 and center $(0, 0)$ oriented counterclockwise;

(b) The line integral $\int F \cdot d\gamma$ is said to be *independent of path* if $\int_{C_1} F \cdot d\gamma = \int_{C_2} F \cdot d\gamma$ for any two paths having the same initial and final points. Give a necessary and sufficient condition on the $C^1$ vector function $F$ on $\mathbf{R}^2$ such that the line integral $\int F \cdot d\gamma$ is independent of path. Justify your answer. (10%)

End of Paper