Entrance Exam of Basic Mathematics

Eight questions with the marks indicated.

1. (10) Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

2. (10) Prove the Schwarz inequality
$$\sum_{i=1}^{n} a_i b_i \le \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}.$$

3. (15) Consider the linear algebraic equations with their perturbations
$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{A}(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{b} + \Delta\mathbf{b}, \tag{1}$$
where the matrix $\mathbf{A} \in R^{n\times n}$ is symmetric, and positive and definite, and $\mathbf{x}, \Delta\mathbf{x}, \mathbf{b}, \Delta\mathbf{b} \in R^n$ are vectors. Prove
$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} \le \frac{\lambda_n}{\lambda_1}\frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}, \tag{2}$$
where $\|\mathbf{x}\|$ is the Euclidean norm of vector $\mathbf{x}$, and λ_n and λ_1 are the maximal and the minimal eigenvalues of $\mathbf{A}$, respectively.

4. (15) Let the matrix $\mathbf{A} \in R^{n\times n}$ and $\mathbf{B} \in R^{n\times n}$ are symmetric, and $\mathbf{B}$ is also positive definite. Prove the eigenvalues of the matrix $\mathbf{B}^{-1}\mathbf{A}$ are all real.

5. (10) Let
$$\ln(x^2 y) + x^3 = y^2 + 2. \tag{3}$$
Find $\frac{dy}{dx}$.

6. (10) Find the original functions for
$$\int \frac{dx}{\sqrt{3 - 2x^2}}. \tag{4}$$

7. (15) Find the sum of the power series
$$\sum_{n=0}^{\infty} \frac{x^n}{n+1} \text{ on } (-1, 1). \tag{5}$$

8. (15) Find the general solutions of the ODE:
$$y^{(4)} - 2y^{(3)} + 5y^{(2)} = 0, \tag{6}$$
where $y^{(n)} = \frac{d^n y}{dx^n}$.

1. An urn contains $n+m$ balls, of which n are red and m are black. They are withdrawn from the urn, one at a time and without replacement. Let Y denote the number of red balls chosen after the first but before the second black ball has been chosen. Number the red balls from 1 to n. Find $E[Y]$. (10%)

2. Let X have pdf $f_X(x) = \frac{2}{9}(x+1)$, $-1 < x < 2$. Find the pdf of $Y = X^2$. (10%)

3. Let X and Y denote the coordinates of a point uniformly chosen in the circle of radius 1 centered at the origin. That is, their joint density is

$$f(x,y) = \frac{1}{\pi}, \qquad x^2+y^2 \le 1$$

Find the joint density function of the polar coordinates $R = (X^2 + Y^2)^{1/2}$ and $\Theta = \tan^{-1} Y/X$. (10%)

4. Suppose that the random variable Y has a binomial distribution with n trials and success probability X, where n is a given constant and X is a uniform$(0,1)$ random variable. (20%)

   (a) Find $E[Y]$ and $\text{Var}(Y)$.

   (b) Find the marginal distribution of Y.

5. Let $X_1, X_2, \ldots$ be iid and $X_{(n)} = \max_{1\le i\le n} X_i$. If X_i is exponential(1), find a sequence a_n so that $X_{(n)} - a_n$ converges in distribution. (10%)

6. Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x|\mu) = e^{-(x-\mu)}$, where $-\infty < \mu < x < \infty$. (20%)

   (a) Show that $X_{(1)} = \min_i X_i$ is complete sufficient statistic.

   (b) Show that $X_{(1)}$ and $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$ are independent.

7. The independent random variables $X_1, \ldots, X_n$ have the common distribution

$$P(X_i \le x|\alpha,\beta) = \begin{cases} 0 & \text{if } x < 0, \\ (x/\beta)^\alpha & \text{if } 0 \le x \le \beta, \\ 1 & \text{if } x > \beta, \end{cases}$$

where the parameters α and β are positive. Find the MLEs of α and β. (10%)

8. Suppose that we have two independent random samples: $X_1, \ldots, X_n$ are exponential(θ), and $Y_1, \ldots, Y_m$ are exponential(μ). Find the LRT of $H_0 : \theta = \mu$ versus $H_1 : \theta \ne \mu$. (10%)

～全卷完～

機率論 [應數系碩士班甲組]

第1-5題16分，第6 題20分。

1. 設有一隨機變數X為有參數λ之指數分佈$\mathcal{E}(\lambda)$。又定義一新的隨機變數Y 為

$$Y = \begin{cases} X & ,\text{若} X \le t_0, \\ t_0 & ,\text{若} X > t_0, \end{cases}$$

其中$t_0 > 0$為一常數。

(i) 試求Y 之分佈函數(d.f.)。

(ii) 設有二分佈函數(d.f.) G 及H, α 為一常數,$0 < \alpha < 1$。
試証$F = \alpha G + (1-\alpha)H$ 仍為一d.f.。

(iii) 試將Y之分佈函數表成如(ii)中二分佈函數之凸組合(convex combination) 的形式，說明對應的α, G, H應為何。

2. 設V為具有均勻分佈$\mathcal{U}(0, 2\pi)$之隨機變數，W為與V獨立且具有指數分佈$\mathcal{E}(\lambda), \lambda > 0$之隨機變數。

(i) 試求

$$X = \sqrt{2W}\,\cos V,\ Y = \sqrt{2W}\,\sin V$$

之分佈，並證明其為獨立。

(ii) 令$Z = X + Y$，試求給定$Z = z$, $z \in R$, X之條件分佈。

3. 設X_1, X_2, X_3 為i.i.d. 之$\mathcal{U}(0,1)$ r.v.'s ,令$Y_1 \le Y_2 \le Y_3$表X_1, X_2, X_3之順序統計量。

(i) 試求$X_1 + X_2$之機率密度函數(p.d.f.)及其分佈函數(d.f.)。

(ii) 試求$P(X_3 \ge X_1 X_2)$。

(iii) 求給定$Y_2 = y, 0 < y < 1, Y_1, Y_3$之聯合分佈。

4. 設Y有$Be(\alpha, \beta)$分佈，而X表投擲一出現正面之機率為Y之銅板n次所得之正面數。

(i) 試求X之(非條件)分佈。

(ii) 試求給定$X = x, Y$之條件分佈，及$E(Y|X = x)$, $Var(Y|X = x)$。

5. 設一袋中有a個白球，b 個黑球。每次從袋中隨機抽出一球，抽出後不放回，直到抽出第一個白球後停止。試求停止抽球時，被抽出黑球個數的期望值。

6. 設$\{X_n, n \ge 1\}$ 為i.i.d.之$\mathcal{N}(0, \sigma^2)$ r.v.'s。令

$$Y_m = \frac{\sum_{i=1}^{k} X_i^2/k}{\sum_{i=k+1}^{k+m} X_i^2/m}$$

(i) 試求Y_m 之分佈。

(ii) 若固定k，試求當$m \to \infty$時，Y_m之極限分佈。

ANSWER *all* OF THE FOLLOWING QUESTIONS, EACH OF WHICH CARRIES 20 OUT OF 100 POINTS.

1. Evaluate the integrals
   (a) $\displaystyle\int_{-1/5}^{1/5} \frac{6\,dx}{x\sqrt{\ln x}}$.
   (b) $\displaystyle\int \frac{dx}{\sqrt{-x^2+4x-1}}$.
2. Identify the stationary points of each function, and determine whether they represent maxima, minima, or inflection points. Confirm your result by determining the sign of the first derivative in the neighborhood of the stationary points.
   (a) $y = -\frac{1}{2}x^6 - 5$.
   (b) $y = \frac{\ln x}{x}$; $x > 0$.
3. Find the interval of convergence of the power series
$$\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n.$$
4. Find the length of the curve
$$y = x^2 - \frac{1}{8}\ln x, \quad 1 \le x \le 4.$$
5. (a) Using the Taylor series of the exponential function $\exp(x) = e^x$ to find a series representation of e^{-1}. Approximate e^{-1} within an error less than $1/10$ by this series.
   (b) Prove that e is an irrational number.

End of Paper

國立中山大學九十四學年度碩士班招生考試試題

科目：數值分析【應數系碩士班乙組】

Please write down all the detail of your computation and answers.

1. (30%) Use (1) Lagrange formula, (2) Neville's method, and (3) Newton divided difference formula to compute the cubic polynomial $p(x)$ interpolating the following data

x	-1	0	1	2
y	-5	-1	3	13

2. (20%) (1) Write down the algorithm of Newton method to solve a root γ of the nonlinear equation $f(x) = 0$.

   (2) If γ is a simple root of $f(x)$ and the initial value is sufficiently close to γ, show that this iteration converges to γ quadratically by the fixed point theorem.

   (3) What happens if γ is a multiple root? Why?

3. (20%) (1) State three pivoting strategies of Gaussian elimination for $n \times n$ matrix A.

   (2) How many comparisons are needed in each strategy?

   (3) Which one has the highest accuracy to solve linear systems?

4. (15%) Write the following matrix A as the PLU decomposition $A = PLU$, where P is a permutation matrix, L lower triangular matrix, and U upper triangular matrix.

$$\begin{pmatrix} 1 & -2 & 3 & 0 \\ 1 & -2 & 3 & 1 \\ 1 & 3 & 2 & -2 \\ 2 & 1 & 3 & -1 \end{pmatrix}$$

5. (15%) Let T be an $n \times n$ matrix and $\mathbf{v}$ be an n dimensional column vector. Prove that the iterative method $\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + \mathbf{v}$ converges if, and only if, the spectral radius $\rho(T) < 1$.

Linear Algebra

1. For which k does

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1 \end{aligned}$$

have no solutions, one solution or infinite many solutions? If it has solutions, find them. (12%)

2. Let U, W be subspaces of a finite dimensional vector space V, $T : V \to V$ be a linear transformation on V, and A be a square matrix and B be an $m \times n$ matrix. Determine each following statement either is true or false. If true, prove it; if false, give a counterexample.

   (a) $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$. (12%)

   (b) $\{A, A^2, A^3, \cdots, A^{n+1}\}$ is linearly independent. (12%)

   (c) If $T^2 = T$, then $V = \ker(T) \oplus \operatorname{range}(T)$. (12%)

   (d) $\operatorname{rank}(B^T B) = \operatorname{rank}(B)$. (12%)

3. Let $W = \{(x, y, z, w) \in R^4 : x - y - 2z = 0, y - z + w = 0\}$. Find the projection matrix on W. (12%)

4. For an $n \times n$ matrix A, define

$$\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{n!}A^n + \cdots.$$

Let $A = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$. Find $\exp(A)$. (14%)

5. Find a Jordan canonical form and a Jordan basis for the matrix

$$\begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & -1 & 0 & 2 \end{bmatrix} \qquad \text{here} \quad i = \sqrt{-1}.$$

(14%)

Entrance Advanced Calculus for the Master program

Twenty points for each problem. Please write down all the detail of your computation and answers. Each carries 20 points.

(1) Evaluate

$$\int_0^1 \frac{x^2+1}{x^4+1}\,dx.$$

(2) (a) (10 points) Find the radius of convergence of

$$\sum_{n=0}^{\infty} \binom{n+2}{2} x^n.$$

(b) (10 points) Evaluate the power series in (a).

(3) Let f be a one-to-one real continuous function on $[0,1]$. Show that f is either strictly decreasing or strictly increasing. Does f have a continuous inverse? Justify your answer.

(4) Let f and g be continuous on $[0,1]$ and differentiable on $(0,1)$.. Suppose that $f(0)=g(0)$ and $f'(x)<g'(x)$ for all $x\in(0,1)$. Prove or disprove: $f(1)<g(1)$.

(5) Let f be a real continuous function on a metric space $\mathbf{S}$. Let $P(f)=\{s\in\mathbf{S}|f(s)\geq 0\}$. Is $P(f)$ a closed subset of $\mathbf{S}$? Justify your answer.

END OF PAPER

科目：複變函數論【應數系碩士班丙組】

ANSWER *all* OF THE FOLLOWING QUESTIONS, EACH OF WHICH CARRIES 20 OUT OF 100 POINTS.

1. Evaluate

$$I = \int_0^{+\infty} \frac{x^{1/2}}{1+x}\, dx.$$

2. Let f be an analytic function in a simply connected regoin R. Let C be the boundary of a rectangle inside R. Prove that the integral

$$\int_C f(w)\, dw = 0,$$

where C is traced counterclockwisely.

3. Prove the fundamental theorem of algebra by applying Rouché theorem to a large circle with $f(z) = z^n$. (The fundamental theorem of algebra says that every non-constant polynomial has at least one zero.)

4. Show that if f is analytic and f' is continuous in a region D, and $|f|$ is constant in D, then f is constant in D. (Hint: Apply Cauchy-Riemann Equations.)

5. Prove that the uniform limit of a sequence of univalent functions is either univalent or constant.

End of Paper