國立中山大學九十三學年度碩士班招生考試試題

科目：基礎數學A (應用數學系碩士班甲組)

共十題，每題10分。答題時，每題都必須寫下題號與詳細步驟。

1. Solve the following system of linear equations

$$\begin{cases} x - 2y + z = 0 \\ 2x + y - z = 2 \\ 4x - 3y + z = 2 \end{cases}.$$

2. Let $\mathbf{A}$ be a $m \times n$ real matrix. Show that $\operatorname{tr}(\mathbf{A}^T\mathbf{A}) = 0$ if and only if $\mathbf{A}$ is a zero matrix.

3. Let $\mathbf{A}$ be a nonsingular matrix, $\mathbf{x}$, $\mathbf{y}$ two column vectors and c a scalar. Show that

$$\begin{vmatrix} \mathbf{A} & \mathbf{y} \\ \mathbf{x}^T & c \end{vmatrix} = |\mathbf{A}|(c - \mathbf{x}^T\mathbf{A}^{-1}\mathbf{y}).$$

4. Let $\mathbf{Q}$ be a positive definite symmetric $n \times n$ matrix. For any column vector $\mathbf{x}$ there holds

$$\frac{(\mathbf{x}^T\mathbf{x})^2}{(\mathbf{x}^T\mathbf{Q}\mathbf{x})(\mathbf{x}^T\mathbf{Q}^{-1}\mathbf{x})} \geq \frac{4aA}{(a+A)^2}$$

where a and A are, respectively, the smallest and largest eigenvalues of $\mathbf{Q}$.

5. Evaluate $\lim_{x\to 0} \frac{\int_0^x \sin(t^3)\,dt}{\int_0^x \sin(t^2)\,dt}$.

6. Use one method of calculus to approximate $\sqrt{10}$ with an error of less than 0.1.

7. The circular disk $(x-2)^2 + y^2 \leq 1$ is rotated around the y-axis. Find the the volume of the doughnut-shaped region generated.

8. Find the radius of convergence and interval of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{2^k(x-4)^k}{\ln(k+2)}.$$

9. Let the curve C be given by $x = t^2$ and $y = t^3$. Calculate the length of the arc from $t = 0$ to $t = 3$.

10. Evaluate

$$\int\int_{x^2+y^2\leq 1} \frac{1}{1+x^2+y^2}\,dxdy.$$

～全卷完～

(1) Let X_i denote the sales to the ith customer of a certain market. Assume X_i's are random and are independent from customer to customer, with $E(X_i) = 25$ and $Var(X_i) = 20, \forall\, i = 1, 2, \cdots$. The number of customers, N, in a day is assumed to be independent of $X_1, X_2, \cdots$ and follow a Poisson distribution with mean of 100 customers, that is

$$P(N = i) = \frac{e^{-100}(100)^i}{i!}, i = 0, 1, 2, \cdots.$$

Find the variance of the sales in one day, namely $Var(S_N)$ where $S_N = X_1 + X_2 + \cdots + X_N$.(15 pts)

(2) A random variable X is said to have a lognormal distribution, if the logarithm of X has a normal distribution. Let $X_1, X_2, \cdots, X_n$ be iid lognormal random variables, thus $Y_i = \ln X_i \sim N(\mu, \sigma^2)$. Use invariance principle of maximum likelihood estimation to find the maximum likelihood estimators of $E(X_i)$ and $Var(X_i)$.(15 pts)

(3) Let (X_1, X_2) be a two dimensional discrete random variable with probability function:

$$P_{X_1,X_2}(x_1, x_2) = \begin{cases} \frac{2}{k(k+1)} & \text{if } x_2 = 1, 2, \cdots, x_1, \text{ and } x_1 = 1, 2, \cdots, k \\ 0 & \text{otherwise,} \end{cases}$$

for a given positive integer k. Find the covariance of X_1 and X_2.(15 pts)

(4) Let $X_1, X_2, \cdots, X_n$ be iid uniform(0,1) and let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ denote the order statistics. Define the range as $R = X_{(n)} - X_{(1)}$, and the midrange as $V = \frac{X_{(1)}+X_{(n)}}{2}$. Find the margianl probability density functions of R and V, respectively.(20 pts)

(5) Let $X_1, X_2, \cdots, X_n$ be iid binomial(k, ϑ) random variables. Find the uniformly minimum variance unbiased estimator (UMVUE) of the parameter $\tau(\vartheta) = k\vartheta(1-\vartheta)^{k-1}$ which is the probability of exactly one success. (15 pts)

(6) Suppose that Θ is a random variable that follows a gamma distribution with the following density function:

$$g(\theta) = \begin{cases} \frac{\theta^{\alpha-1}e^{-\theta}}{\Gamma(\alpha)} & \text{for } \theta \geq 0 \\ 0 & \text{for } \theta < 0, \end{cases}$$

where α is an integer, and suppose that conditional on Θ, X follows a Poisson distribution with parameter Θ. Find the unconditional distribution of $\alpha + X$ and establish the relationship between $P(\Theta \leq \vartheta)$ and $P(X \geq \alpha | \Theta = \vartheta)$.(20 pts)

國立中山大學九十三學年度碩士班招生考試試題

科目：機率論 [應數系碩士班甲組]

第1-5題16分，第6 題20分。

1. 自區間$\Omega = [0,1]$中隨機地取一個點。對Ω上之一Borel 集合B,令$P(B) = B$之長度=此點會落在B之機率。又令$C = \Omega \setminus A$ 為Cantor集合，其中$A = \bigcup_{n=1}^{\infty} A_n$, 而$A_1 = (\frac{1}{3}, \frac{2}{3})$,$A_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$， 對一般$n \geq 3$, A_n為集合$(A_1 \cup \cdots \cup A_{n-1})^c$之$2^{n-1}$個子區間的中間$\frac{1}{3}$區間的聯集。 試証$P(C) = 0$。

2. 設$X_1, X_2, \cdots, X_n$ 為i.i.d.之隨機變數，且$Y = \min\{X_1, \cdots, X_n\}$ 有參數為λ 之指數分佈。

   (i) 試求 X_i 之分佈，並說明為那一常見分佈，參數為何。

   (ii) 又若令$Y_1 = X_1 - X_2$, $Y_2 = X_2$, 試求Y_1, Y_2之聯合分佈，與Y_1之邊際分佈，並說明Y_1之分佈為那一常見分佈，參數為何。

3. 設給定$X = x, 0 < x < 1, Y$有幾何分佈, 參數為x, 又設X有$Be(\alpha, \beta)$分佈 。

   (i) 試求Y之(非條件)分佈。

   (ii) 若$\alpha = \beta = 1$, 試求$P(X > \frac{1}{2} | Y = y)$, $y = 0, 1, \cdots$。

4. 設有一射手在打靶練習時，其射擊點之分佈大致如(X, Y)，其中X, Y 為二獨立之隨機變數，且以$\mathcal{N}(0, \sigma^2)$ 為其共同分佈。

   (i) 試估計此射手所射擊之點，會落在以靶正中央為圓心，半徑為$r, r > 0$ 之圓內之機率。

   (ii) 試推估此射手射擊點與靶心的平均距離。

5. 某工廠生產螺絲釘, 每個螺絲釘會是不良品的機率設為$p = 0.015$。

   (i) 若螺絲釘每100 個裝一盒出售。試求每盒中不良品數，至多只有1個的機率並給出其近似值。

   (ii) 若欲一盒中良品數有100 個以上的機率至少是0.8, 則一盒中須放多少個螺絲釘?

6. 設$U_1, \cdots, U_n$為一組由 $\mathcal{U}(0,1)$ 分佈所產生之隨機樣本。令$G_n = (U_1 U_2 \cdots U_n)^{1/n}$。

   (i) 試求$V_1 = -\log(U_1)$之分佈函數。

   (ii) 試求$W_n = -\log(G_n)$之特徵函數，並說明W_n之分佈為那一常見分佈，參數為何。

   (iii) 試證$n \to \infty$時, G_n 機率收斂至G, G為一常數r.v.,並給出此常數。

Linear Algebra （注意：每個題目需證明或說明清楚，只填答案不計分。）

Let $\mathbb{R}$ be the set of all real numbers and $M_{m\times n}(\mathbb{R})$ be the set of all m×n matrices over $\mathbb{R}$.

1. Let $V=\{(x,y,z)\in\mathbb{R}^3 : x+y+z=0\}$. Define a vector addition "+" and scalar multiplication "*" for V such that $(V,+,*)$ is a vector space over $\mathbb{R}$. (15%)

2. Suppose (a,b,c) and (d,e,f) are linearly independent on $\mathbb{R}^3$. Prove that (x,y,z) is a linear combination of (a,b,c) and (d,e,f) if and only if $\det\begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix}=0$. (15%)

3. Suppose T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ and the dimension of the image of T is k. To show that there exist ordered bases α and β such that the matrix representation of T with respect to α and β is a diagonal matrix $D=(d_{ij})$ where $d_{ii}=i$ for $i\leq k$, and $d_{ii}=0$ for $n\geq i>k$. (15%)

4. Suppose $A\in M_{n\times n}(\mathbb{R})$. Prove that there exist an orthogonal matrix Q and a lower-triangular matrix L such that A=LQ. (15%)

5. Determine each following statement either is true or false. If true, prove it; if false, give a counterexample. (8%×5)
   (a) Suppose $A\in M_{n\times n}(\mathbb{R})$ and there exists $k\in\{2,3\}$ such that $A^k=A$. Then A is diagonalizable.
   (b) Suppose $A, B\in M_{n\times n}(\mathbb{R})$. If A and B have same minimal polynomial then they have the same Jordan canonical from.
   (c) If $r\neq s$ then the matrix $\begin{pmatrix} r & s & s & s \\ s & r & s & s \\ s & s & r & s \\ s & s & s & r \end{pmatrix}$ has at most two eigenvalues.
   (d) If T is a one-to-one linear operator on a vector space V then T is onto.
   (e) Suppose m≠n and $A\in M_{m\times n}(\mathbb{R})$. If the linear system $A\vec{x}=\vec{b}$ has a solution, then the number of the solutions for the linear system is infinite.

科目：微積分（應用數學系碩士班乙組）

1. 設 $g(x)$ 為一連續函數，$g(1)=2$，$g(2)=-1$，
$\int_1^2 g(t)\,dt=3$，且令 $F(x)=\int_{x^3}^{2} g(xt)\,dt$，$x\in R$。
求 $F'(1)$。 (12分)

2. 設 $f(x)$ 為在 $[0,1]$ 區間可二次微分之函數且 $|f''(x)|\le K$，
$K>0$。証明：如果 $f(x)$ 在 $(0,1)$ 內有極大值，則
$|f'(0)|+|f'(1)|\le K$。 (16分)

3. 求從橢圓 $x^2+xy+y^2=3$ 到原點的最近和最遠的距離。 (16分)

4. (a) 求 $\int_0^1\int_y^1 \frac{1}{1+x^4}\,dx\,dy$

(b) 求 $\iint_D (x+y)\,dx\,dy$，其中 D 表示由 $y=2x+3$ 及 $y=x^2$ 所圍区域。

(c) 用任何積分方法証明：半徑為 R 的半球，其体積為 $\frac{2}{3}\pi R^3$。

(以上每小題 8 分，第 4 題共 24分)

5. 求 $\lim_{x\to 0}\frac{\sin^2 x\cos(2x)+\tan x}{2\sec^2 x\cos(3x)-2}$ (15分)

6. 設 $f_{12}(x,y)=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f(x,y)\right)$，$f_{21}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f(x,y)\right)$。

若 $f(x,y)=\begin{cases}\frac{xy(y^2-x^2)}{x^2+y^2}, & (x,y)\ne(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$ 求 $f_{12}(0,0)$ 及 $f_{21}(0,0)$ (17分)

Entrance Exam of Numerical Analysis for the Master program

Twenty points for each problem. Please write down all the detail of your computation and answers.

1. Describe convergence and stability for numerical methods, give relations between them, and provide examples to explain your answers.

2. Suppose that there exists a root of $f(x) = 0$, and $0 \le m \le f'(x) \le M$. Prove that

$$x_{n+1} = x_n - \lambda f(x_n)$$

yields the convergent sequence $\{x_n\}$ to the root for arbitrary $x_0 \in (-\infty, \infty)$ and $0 < \lambda < 2/M$.

3. Give the trapezoidal and midpoint rules for the integral, $I = \int_a^b f(x)dx$. Show that when $f''(x) \ge 0$ on $[a, b]$, the approximate integrations by the trapezoidal and midpoint rules are the upper and lower bounds of I, respectively.

4. Derive the error bounds for the trapezoidal rule in two dimensions:

$$\int_0^h \int_0^k g(x,y)dxdy \approx \frac{hk}{4}(g(0,0) + g(h,0) + g(0,k) + g(h,k)).$$

5. Given an original image $\{\phi_{ij}\}$ and other two images, $\{u_{ij}\}$ and $\{v_{ij}\}$, consisting of 256 × 256 pixels with 256 greyness levels. Form a linear combination

$$\{w_{ij}\} = \alpha\{u_{ij}\} + \beta\{v_{ij}\}.$$

Provide a numerical method to seek the parameters α and β such that the combined image $\{w_{ij}\}$ is best approximate to the original image $\{\phi_{ij}\}$.

Entrance Advanced Calculus for the Master program

＊Twenty points for each problem. Please write down all the detail of your computation and answers.

1. Prove

$$\{\int_a^b wf(x)g(x)dx\}^2 \leq \{\int_a^b wf^2(x)dx\}\{\int_a^b wg^2(x)dx\},$$

where $w \geq 0$ on $[a, b]$.

2. Determine whether the following singular definite integral is convergent or not,

$$\int_e^{10} \frac{dx}{x \ln x \ln(\ln x)}.$$

3. Find the sum of the limit

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right).$$

4. Evaluate the integral on three dimensions,

$$I = \int\int\int_\Omega (x^2+y^2)dv,$$

where the integration region Ω is surrounded by the rotating parabola $z = x^2 + y^2$ and the plane $z = 1$.

5. Prove that the functions $r^n \cos n\theta$ and $r^n \sin n\theta$ satisfy the Laplace equation,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

where (r, θ) are the polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$.

科目：複變函數論（應數系碩士班丙組）

In the following, $\mathbf{C}$ is the set of complex numbers, $\mathbb{R}$ is the set of real numbers, and i is a square root of -1.

Problem 1 Let $C = \{z \in \mathbf{C} : \frac{|z-i|}{|z+2|} = 2\}$. Prove that C is a circle, and find its center and radius. (10 %)

Problem 2 Let $f(z) = |1 - \cos z|$ for $z \in \mathbf{C}$.

(i) Prove that f is differentiable at 0. (10 %)

(ii) Prove that f is not analytic at 0. (10 %)

Problem 3 Let $\gamma(\theta) = 2e^{i\theta}$ for $0 \le \theta \le 4\pi$. Evaluate : $\int_\gamma \frac{1}{(2z+1)(z-3)}\, dz$ (10 %)

Problem 4 Let $\gamma(\theta) = 3e^{i\theta}$ for $0 \le \theta \le 2\pi$. Evaluate $\int_\gamma \frac{|dz|}{|2z-1|^2}$. (15 %)

Problem 5 Let $\Omega = \{z \in \mathbf{C} : 0 < |z| < 3\}$, and let $f : \Omega \longrightarrow \mathbf{C}$ be an analytic function. Assume that $|f(z)| \le 3$ for $|z| = 1$, and that $|f(z)| \le 12$ for $|z| = 2$. Prove that $|f(z)| \le 3|z|^2$ for $1 \le |z| \le 2$. (15 %)

Problem 6 Let T be a Möbius transformation. Assume that $T(1) = i$, $T(\frac{i}{2}) = 2$, and $|T(z)| = 1$ for all $z \in \mathbf{C}$ with $|z| = 1$. Find the transformation T. (15 %)

Problem 7 Let $f : \{z \in \mathbf{C} : |z| < 2\} \longrightarrow \{z \in \mathbf{C} : |z| < 1\}$ be an analytic function with $f(i) = \frac{1}{2}$. Prove that if $|z| < 2$, then

$$\frac{|2f(z) - 1|}{|2 - f(z)|} \le \frac{2|z - i|}{|4 + iz|}. \quad (15\ \%)$$