國立中山大學九十二學年度碩士班招生考試試題

科目：基礎數學 A（應用數學系碩士班　甲組）

1. Consider the quadratic surface S: $3x^2 - 2xy + 2xz + 3y^2 - 2yz + 3z^2 = 1$. Answer the following questions.

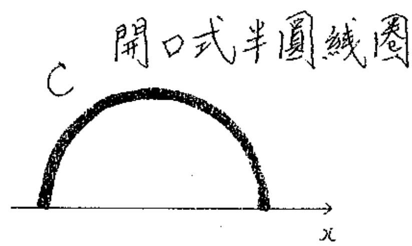
(a) If we write the quadratic form as $(x, y, z)A(x, y, z)^T = 1$, where (x, y, z) is a row vector, $(x, y, z)^T$ is the transpose of (x, y, z) and A is a 3×3 symmetric matrix. Find the matrix A. (5 points)
(b) Find all the eigenvalues of A. (5 points)
(c) Find the corresponding eigenvectors of A. (5 points)
(d) Find an orthogonal matrix P such that P^TAP is a diagonal matrix with the eigenvalues of A on the diagonal. (5 points)
(e) Consider the problem of finding the distance from the surface S to the origin $(0, 0, 0)$. The problem is equivalent to finding the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the condition $3x^2 - 2xy + 2xz + 3y^2 - 2yz + 3z^2 = 1$. Show that the problem is the same as (b) that if the eigenvalues in (b) are λ_1, λ_2, λ_3, then the solution of this problem is $\min\{{\lambda_1}^{-1/2}, {\lambda_2}^{-1/2}, {\lambda_3}^{-1/2}\}$. (15 points)
(f) Give a geometric explanation from the point of view of (d). (5 points)

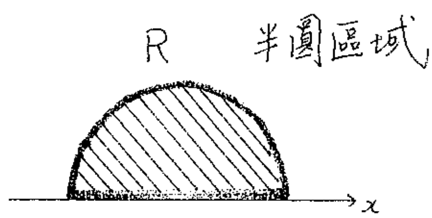
2. Consider the function $f(x) = e^{-\frac{1}{x^2}}$, if $x \neq 0$, and $f(x) = 0$ if $x = 0$. Answer the following questions.

(a) Find $f'(1)$. (2 points)
(b) Prove that $f'(0) = 0$. (5 points)
(c) Prove that $f''(0) = 0$. (5 points)
(d) Prove that $f^{(n)}(0) = 0$, $\forall n \geq 1$. (8 points)

3. Consider a semi-circle arc C and a semi-circle area R (see the figures below). Answer the following questions.

(a) If C is homogeneous, find the center of mass of C. (10 points)
(b) If the density at (x, y) in R is proportional to $x^2 + y^2$, find the center of mass of R. (10 points)





國立中山大學九十二學年度碩士班招生考試試題

科目：基礎數學 A

4. Answer the following four questions.

(a) Find the integral $\int_0^1$ x lnx dx. (5 points)
(b) Find the value $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}$. (5 points)
(c) Describe a method of how to estimate the integral $\int_0^{1/3} e^{-\frac{1}{2}x^2}dx$ to a certain accuracy. (5 points)
(d) Describe how to use Simpson's rule to estimate ln2. (you need to find a definite integral that represents ln2). (5 points)

國立中山大學九十二學年度碩士班招生考試試題

科目： 數理統計　　(應用數學系碩士班甲組)

(1) Let $X_1, \cdots, X_n$ be a sample from $U(0,1)$ distribution and denote the order statistics by $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$. Find the mean and variance of the kth order statistic $X_{(k)}$. (10 pts)

(2) Suppose that $X_1, X_2, \cdots, X_n$ are iid random variables with density function $f(x|\vartheta)$. Find the maximum likelihood estimates of ϑ for the following two cases.

(a) $f(x|\vartheta) = e^{-(x-\vartheta)}, x \geq \vartheta; = 0$, otherwise. (10 pts)

(b) $f(x|\vartheta) = \frac{1}{2}e^{-|x-\vartheta|}, -\infty < x < \infty$ for $n = 2m+1$. (10 pts)

(3) Let $X_1, X_2, \cdots, X_n$ be iid sample with the following density function (a Rayleigh distribution) $f(x|\vartheta) = \frac{x}{\vartheta^2}e^{-\frac{x^2}{2\vartheta^2}}, x \geq 0$ with parameter $\vartheta > 0$. Find the Cramer-Rao lower bound on the variance of any unbiased estimate of ϑ. (10 pts)

(4) Suppose X is a gamma random variable with probability density function $f(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha}e^{-x/\beta}, x \geq 0$. Using moment generating function, show that as $\alpha \to \infty$, $\frac{X-\alpha\beta}{\sqrt{\alpha}\beta}$ tends to standard normal distribution. (10pts)

(5) (a) Let X_1, X_2 be iid $U(1,\beta)$ and set $Y = X_1X_2$. Find the probability density function of Y. (10 pts)

(b) Suppose that X is a $U(0,1)$ and set $Y = \log X$. If $Y_j, j = 1, 2, \cdots, n$ are independent random variables distributed as the random variable Y, find the probability density function of $\sum_{j=1}^n Y_j$. (10 pts)

(6) Let $X_1, \cdots, X_{30}$ be gamma random variables with parameter $\alpha = 10$ and β unknown (see problem 4 for the density function). Construct the most powerful test of the hypothesis $H_0 : \beta = 2$ against the alternative $H_a : \beta =$ [illegible], at level of significance 0.05. (10 pts)

(7) Let $X_1, \cdots, X_n$ be iid random variables from $N(\mu, \sigma^2)$ with σ^2 unknown and μ known. Find a UMVUE (uniformly minimum variance unbiased estimate) of σ. (10 pts)

(8) Let T_1 and T_2 be sufficient statistics for ϑ, and suppose that $T_2 = g(T_1)$ for some function g. Let U be an unbiased estimate of ϑ, and let $V_1 = E(U|T_1), V_2 = E(V_1|T_2)$. Show that $Var(V_2) \leq Var(V_1)$. (10 pts)

共七題，滿分100分。答題時，每題都必須寫下題號與步驟。

1. The probability of winning on a single toss of the dice is p. A starts, and if he fails, he passes the dice to B, who then attempts to win on her toss. They continue tossing the dice back and forth until one of them wins. What are their respective probabilities of winning? (10分)

2. Urn 1 has five white and seven black balls. Urn 2 has three white and twelve black balls. We flip a fair coin. If the outcome is head, then a ball from urn 2 is selected. Suppose that a white ball is selected. What is the probability that the coin landed tails? (10分)

3. Suppose the density of X is given by

$$f(x) = \begin{cases} \frac{1}{4}xe^{-x/2}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the moment generating function and find $E[X]$, and $\mathrm{Var}(X)$ by the moment generating function. (20分)

4. Evaluate (15分)

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

5. Let X_1, X_2, X_3 and X_4 be independent continuous random variables with a common distribution function F and let

$$p = P\{X_1 < X_2 > X_3 < X_4\}$$

(a) Argue that the value of p is the same for all continuous distribution functions F. (7分)

(b) Find p by integrating the joint density function over the appropriate region. (7分)

(c) Find p by using the fact that all 4! possible orderings of $X_1, \ldots, X_4$ are equally likely. (6分)

6. If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X, given that $X + Y = n$. (15分)

7. Let X be a continuous nonnegative random variable with density function f, and let $Y = X^n$. Find f_Y, the probability density function of Y. (10分)

～全卷完～

國立中山大學九十二學年度碩士班招生考試試題

科目：線性代數　(應用數學系碩士班 乙，丙組)　

1. (10分) 請證明Cramer's rule, i.e, let

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i, \quad i = 1, 2, \cdots, n$$

and suppose that the coefficient matrix $A = (a_{ij})$ of the above system of linear equations is invertible. Prove that $x_i = \frac{\det(A_i)}{\det A}$, where A_i is the matrix obtained from A by replacing the i-th column of A by $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

2. (30分) Let $\alpha_1 = (-1, 1, 0, 1, 3)$, $\alpha_2 = (1, 1, 1, 0, 5)$, $\alpha_3 = (1, 3, 2, 1, 7)$, $\alpha_4 = (0, 0, 0, 0, 1)$; $\beta_1 = (0, 3, 0, 1, 0)$, $\beta_2 = (-3, 1, -1, 2, 2)$, $\beta_3 = (0, -4, 1, 0, 1)$, $\beta_4 = (-1, 0, 0, 1, 1) \in \mathbb{R}^5$. Suppose that M_1 is (linear) spanned by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and M_2 is spanned by $\beta_1, \beta_2, \beta_3, \beta_4$. Find the dimension and basis of the spaces M_1; M_2; $M_1 + M_2$ and $M_1 \cap M_2$.

3. (15分) Let $A, B \in M_{m\times n}(\mathbb{R})$. Prove that $\text{rank}(A + B) \le \text{rank}A +$ $\text{rank}B$, where $\text{rank}X$ means the rank of $X \in M_{m\times n}(\mathbb{R})$.

4. (15分) 設L_1，L_2 為過原點之直線，且與x 軸之夾角分別為$\frac{\pi}{3}$ 與$\frac{\pi}{5}$，問：先對L_1 鏡射，再對L_2 鏡射之合成為旋轉或鏡射？若旋轉，請找出其旋轉角；若鏡射，請找出其鏡射軸。

5. (15分) 令T 表在$\mathbb{R}^3$ 中以$(1, 2, 2)$ 為軸, 逆時針方向旋轉90° 之線性變換, 試求T 在標準基底e_1, e_2, e_3 之下的表現矩陣.

6. (15分) 求解$X_1(t), X_2(t), X_3(t)$ 滿足

$$\begin{aligned} X_1' &= X_1 - X_2 - X_3 \\ X_2' &= -X_1 + X_2 - X_3 \\ X_3' &= -X_1 - X_2 + X_3 \end{aligned}$$

where X_i' denote the derivative of X_i.

～全卷完～

國立中山大學九十二學年度碩士班招生考試試題

科目： 微積分　　(應用數學系碩士班乙組)　　共 1 頁 第 1 頁

填充題(40%)(不須要計算過程)

1. $\int \frac{1}{e^{2x}+e^{-2x}}\,dx =$ ______________.

2. If $f(x) = \int_2^x \frac{dt}{\sqrt{1+t^3}}$, then $(f^{-1})'(0) =$ ______________.

3. The value of $\lim_{x\to\infty} \frac{\int_\pi^{x^2} t\ln t\,dt}{x^4}$ is ______________.

4. The general solution of $x\frac{dy}{dx} - 2y = x^2$ is ______________.

5. Let C be the circle $x^2+y^2=16$, then

$$\oint_C (y+\sqrt{1+x^2})\,dx + (4x - e^{\sin y})\,dy = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}.$$

計算証明題(60%)(請詳述過程)

1. Prove that $|\cos x - \cos y| \le |x-y|$ for all x and y.

2. Find the area bounded by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

3. A manufacturer drills a hole through the center of a metal sphere of radius 10 cm, the hole has a radius of 6 cm. What is the volume of the resulting ring?

4. Find the positive values of $p \in \mathbb{R}$ for which

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$$

converges.

5. Find the highest point on the curve of the intersection of $x^2+y^2+z^2 = 36$ and $2x+y-z=2$.

6. Let $z = x+iy$ be the complex variable. Consider the function u defined in $\mathbb{R}^2$ by

$$u(x,y) = \begin{cases} \operatorname{Re} e^{-\frac{1}{z^4}}, & \text{for} \quad (x,y) \ne (0,0), \\ 0, & \text{for} \quad (x,y) = (0,0). \end{cases}$$

Show that

(1) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ everywhere in $\mathbb{R}^2$.

(2) u is not continuous at the origin.

Twenty points for each problem. Please write down all the detail of your computation and answers.

1. (1)Why a subtraction of two close floating point numbers is unstable? Please give an example to illustrate it.

   (2)For very small number $\varepsilon \approx 0$, how to compute $\sqrt[3]{x+\varepsilon} - \sqrt[3]{x}$ and $\sin(x+\varepsilon) - \sin x$ to avoid unstable substraction.

2. (1)Under what conditions will a fixed point iteration

$$x_n = g(x_{n-1}), n = 1, 2, \cdots$$

   converge?

   (2)Let $a > 0$. Show that the sequence

$$x_n = \frac{x_{n-1}}{2} + \frac{a}{2x_{n-1}}, n = 1, 2, \cdots.$$

   converges to $\sqrt{a}$ for all $x_0 > 0$. What happens if $x_0 < 0$?

3. Derive the composite trapezoidal rule for numerical integration with error formula.

4. Let A be an $n \times n$ nonsingular matrix and B be an $n \times m$ matrix. State the fast numerical method to compute (1) determinant of A, (2) $A^{-1}B$. How many arithmetic operations are needed in (1), and what is the minimal memory needed in (2)?

5. Let A be an $n \times n$ nonsingular matrix, $\mathbf{x}$ and $\hat{\mathbf{x}}$ be the exact and numerical solutions of linear system $A\mathbf{x} = \mathbf{b}$ respectively, and the residual $\mathbf{r} = \mathbf{b} - A\hat{\mathbf{x}}$. Show that

$$\frac{1}{\kappa(A)} \frac{||\mathbf{r}||}{||\mathbf{b}||} \leq \frac{||\hat{\mathbf{x}} - \mathbf{x}||}{||\mathbf{x}||} \leq \kappa(A) \frac{||\mathbf{r}||}{||\mathbf{b}||},$$

   where $\kappa(A)$ is the condition number of A. Interpret the meaning of this error analysis and explain why it is important.

國立中山大學九十二學年度碩士班招生考試試題

科目： 高等微積分【應數系碩士班丙組】

ANSWER ALL 4 QUESTIONS, EACH OF WHICH CARRIES 25 POINTS.

(1) (a) Let $\{x_n\}$ be a Cauchy sequence in a metric space $\mathbf{X}$. Suppose that $\{x_n\}$ has a convergent subsequence in $\mathbf{X}$. Prove or disprove: $\{x_n\}$ itself converges in $\mathbf{X}$. (15%)

(b) State and prove the Bolzano-Weierstrass Theorem for real numbers. (10%)

(2) Let $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ for $x > 0$.

(a) Show that $\Gamma(x+1) = x\Gamma(x)$, $x > 0$. (10%)

(b) Find $\Gamma(3.5)$. (15%)

(3) This problem is about the Fundamental Theorem of Calculus. Let f be a continuous function on the finite closed interval $[a, b]$.

(a) Show that $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, $x \in (a, b)$. (10%)

(b) If $F' = f$, show that $\int_a^b f(t)\,dt = F(b) - F(a)$. (15%)

(4) (a) Find the limit $\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2}$ if it exists. (10%)

(b) Let $f(x, y) = \frac{xy}{x^2+y^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Is f differentiable at $(0, 0)$? (15%)

END OF PAPER

複變函數論

Answer each of the following problems. Each of them carries 20 %. Show details of the your work. Below i stands for $\sqrt{-1}$ and z stands for a complex number.

(1) (a) Show that $g(z) = |z|$ is not differentiable at any point. (10%)

(b) Find a harmonic conjugate for the function $u(x,y) = x^3 - 3x^2y + y$.(10%)

(2) Let $f = u + iv$ be an entire function such that $au + bv > c$ for some $a, b, c \in \mathbf{R}$. Show that f must be a constant. (Hint: Find a suitable nonconstant entire function g such that $e^{g(f(z))}$ is constant.) (20%)

(3) Evaluate the following contour integrals where the contour $\gamma = \{z \in \mathbf{C} : |z| = 7\}$, positively oriented

(a) $\int_\gamma \frac{e^{2z}}{(z-1)^2}\,dz$; (10%)

(b) $\int_\gamma \tan 2z\,dz$. (10%)

(4) Derive the Laurent series of the functions below:

(a) $f(z) = \frac{z}{z^2-6z+5}$ in the annulus $1 < |z| < 5$; (10%)

(b) $e^{-1/(z-1)}$ in the domain $|z-1| > 0$. (10%)

(5) Determine if there is any analytic function f defined on the disk $B = \{z \in \mathbf{C} : |z| < 2\}$ satisfying each of the conditions below. If yes, find the function. If not, explain why.

(a) f maps B to the half plane below the line $\mathrm{Im}(z) = \mathrm{Re}(z) + 1$. (10%)

(b) $f(\frac{1}{n}) = \frac{n}{4-n}$. (10%)

End of Paper