(一)

(a) Suppose that $y_i > 0$, $x_i > 0$ for $i = 1, 2, 3$. Prove that the matrix

$$\begin{pmatrix} x_2+x_1+y_1 & -x_2 & 0 \\ -x_2 & x_3+x_2+y_2 & -x_3 \\ 0 & -x_3 & x_3+y_3 \end{pmatrix}$$

is positive definite. (10分)

(b) Prove the general case that the matrix

$$\begin{pmatrix} z_1 & -x_2 & 0 & \cdots & 0 \\ -x_2 & z_2 & -x_3 & \cdots & 0 \\ 0 & -x_3 & z_3 & \cdot & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & z_m \end{pmatrix}$$

(15分)

is positive definite if $z_i = x_{i+1} + x_i + y_i$, $y_i > 0$, $x_i > 0$ $\forall$ $i = 1, 2, \cdots, m$, $x_{m+1} = 0$

(c) Define $f(P_1, P_2, P_3) = \sum_{j=1}^{3} \{ a_j \ln(P_{j-1} - P_j) + b_j \ln P_j + c_j \ln(1 - P_j) \}$

where $a_j, b_j, c_j > 0$ for $j = 1, 2, 3$. $P_0 = 1$, $1 > P_1 > P_2 > P_3 > 0$.

Let $D_{ij} = \partial^2 f(P_1, P_2, P_3) / \partial P_i \partial P_j$, $i, j = 1, 2, 3$, be the second order derivatives.

and denote

$$-D = [-D_{ij}]$$

(15分)

Show by (a) that $-D$ is positive definite.

(二). A rectangular box is to be inscribed in the cone $z = 9 - \sqrt{x^2 + y^2}$, $z \geq 0$. Find the dimensions for the box that maximize its volume. (15分)

(三) Given $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$. Find (a) $\int_0^\infty x^2 e^{-x^2}dx$ (b) $\int_0^\infty x^n e^{-x^2}dx$ where n is a positive integer. ((a) 5分, (b) 10分).

(四). Suppose $f(x)=\begin{cases} e^{-\frac{1}{x^2}} & x\neq 0 \\ 0 & x=0 \end{cases}$.

(a) Show that $f'(0)=0, f''(0)=0$. (5分)

(b) Show that $f^{(n)}(0)=0$ if n is a positive integer. (10分)

(五) (a) Find the integral $\int_0^1\int_{\sqrt{x}}^1 \frac{dy}{\sqrt{1+y^3}}dx$. (7分)

(b) Find $\iiint_R z\,dxdydz$ where R is the solid tetrahedron (四面体) formed by (0,0,0). (1,0,0), (0,2,0). (0,0,3). (8分)

國立中山大學九十學年度碩博士班招生考試試題

科目：數理統計【應數系碩士班】甲組

(1) Consider a random sample $X_1, X_2, \cdots, X_n$ from a Poisson distribution with parameter λ. Let $\vartheta = e^{-\lambda}$.

(a) Find the Cramer-Rao lower bound (CRLB) for the unbiased estimators of ϑ. (5pt)

(b) Show that $\tilde{\vartheta} = ((n-1)/n)^{\sum_{i=1}^{n} X_i}$ is an unbiased estimator of ϑ. Also find $Var(\tilde{\vartheta})$ and compare to the CRLB of (a). (15pt)

(2) Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables with continuous distribution function F and let x_p be the p-th quantile, that is $F(x_p) = p$.

(a) Define the random variables $W_j, j = 1, 2, \cdots, n$ as follows

$$W_j = \begin{cases} 1, & \text{if } X_j \le x_p \\ 0, & \text{if } X_j > x_p \end{cases} \quad j = 1, 2, \cdots, n.$$

Find the distributions of W_i's and show that $P(X_{(i)} \le x_p) = \sum_{k=i}^{n} \binom{n}{k} p^k q^{n-k}$, where $X_{(i)}$ denotes the ith order statistic (e.g. $X_{(1)}$ denotes the minimum of $X_1, \cdots X_n$.). (10pt)

(b) Show that $P(X_{(i)} \le x_p \le X_{(j)}) = \sum_{k=i}^{j-1} \binom{n}{k} p^k q^{n-k}$, for $1 \le i < j \le n$, where $q = 1 - p$. (10pt)

(3) Let X_1, X_2 be independent random variables distributed as $U(0,1)$. Find the probability density function and distribution function of the random variable $Y = 2X_1 + X_2$. (15pt)

(4) Let $\hat{\vartheta}$ be an estimator of ϑ with $E(\hat{\vartheta}^2) < \infty$ for all ϑ. Suppose that T is sufficient for ϑ and let $\tilde{\vartheta} = E(\hat{\vartheta}|T)$.

(a) Prove that $E(\tilde{\vartheta} - \vartheta)^2 \le E(\hat{\vartheta} - \vartheta)^2$, for all ϑ. Also show that this equality is strict unless $\hat{\vartheta} = \tilde{\vartheta}$. (10pt)

(b) Based on the result of (a), explain how an estimator can be improved, if it is not a function of a sufficient estimator. Also explain why sufficiency of T is necessary in this approach. (10pt)

(5) Let X and Y have the joint density $f(x, y) = e^{-y}$, $0 \le x \le y$. Find $E(X|Y = y)$ and $E(Y|X = x)$. (10pt)

(6) Suppose that in the population of twins, males (M) and females (F) are equally likely to occur and that the probability that twins are identical is α. If twins are not identical, their genes are independent. Suppose that n twins are sampled. It is found that n_1 are MM, n_2 are FF, and n_3 are MF, but it is not know which twins are identical. Find the maximum likelihood estimator of α. (15pt)

國立中山大學九十學年度碩博士班招生考試試題

科目：機率論【應數系碩士班】甲組　　　　　　　　　　　　共　頁第　頁

第1-4 題每題16 分，第5-6題18 分

1. 一航空公司根據過去的記錄，一般有90%的訂位旅客會前來搭機，旅客彼此間均爲獨立行動。現在某一班機可搭載300位旅客。

   (a) 若航空公司接受了324位旅客的訂位，則此班飛機會發生超額訂位的機率爲何？

   (b) 根據上述過去經驗，則有300座位的飛機，估計最多只能接受多少訂位，使此班飛機會發生超額訂位的機率不會超過2.5%？

   (c) 若324位旅客中已有100位旅客旅客出現劃位了，則此班飛機會發生超額訂位的機率爲何？

2. 設X, Y, Z 爲i.i.d. $\mathcal{E}(1)$ 之r.v.'s

   (a) 試求$P(X \le 2Y$且$X \le 2Z)$。

   (b) W 爲一整數值之r.v. 且$W = m$ 若$m \le X \le m+1$, 其中m 爲一非負整數, 求W 之分佈。

3. 設X, Y 爲二隨機變數，其聯合分佈的機率密度函數(p.d.f.)如下

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{x(y-x)}}e^{-y/2}, \qquad (0 < x < y)$$

   (a) 試求Y之邊際分佈。

   (b) 試求給定$Y = y$時，X之條件分佈。

   (c) 試求$E(X|Y = 1)$。

4. (a) 設X_1, X_2爲二獨立的r.v.'s，$S = X_1 + X_2$。若X_i有$\Gamma(\alpha_i, \beta)$分佈，則S有$\Gamma(\alpha_1 + \alpha_2, \beta)$分佈。

   (b) 設Y_n有$\Gamma(n, \beta)$分佈，以F_n 表$(Y_n/\beta - n)/\sqrt{n}$ 之累積機率函數(c.d.f.)。問$n \to \infty$時 F_n會弱收斂至何d.f. F?

5. 設U, V 爲二獨立之$\mathcal{N}(0,1)$ 分佈, 令

$$Z = \gamma U + \sqrt{1-\gamma^2}V, \qquad |\gamma| < 1$$

   (a) 求Z 之特徵函數$\phi(t) = E(e^{itZ}), t \in R$, 並說明其分佈爲何。

   (b) 求 $X = \mu_1 + \sigma_1 U$ 及 $Y = \mu_2 + \sigma_2 Z$ 之聯合分佈, 其中$\sigma_1, \sigma_2 > 0$，並求其相關係數$\rho(X, Y)$。

6. 令$X_1 \le X_2 \le \cdots \le X_n$ 爲一組連續型的隨機樣本，其共同分佈的累積機率函數(c.d.f.)爲F. $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ 爲$\{X_i, 1 \le i \le n\}$ 之順序統計量(order statistics)。

   (a) 試證$F(X)$有$U[0,1]$分佈。

   (b) 試求$(F(X_{(1)}), F(X_{(n)}))$ 之聯合分佈之機率密度函數(p.d.f.)。

國立中山大學九十學年度碩博士班招生考試試題

科目：線性代數【應數系碩士班】乙.丙組 共 / 頁 第 / 頁

Do all problems in detail.

(1) Let T be a linear mapping on a finite-dimensional vector space V. Prove or disprove: $V = Ker(T) \oplus Im(T)$. [15%]

(2) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $T(x, y, z) = (x - 2y + z, x - 4y - z, x - y + z)$. Find the matrix representation of T with respect to the basis $E = \{\mathbf{e}_1 = (1, 0, -1), \mathbf{e}_2 = (1, 1, 0), \mathbf{e}_3 = (1, 1, -1)\}$. [15%]

(3) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ with $T(x, y, z) = (3x + y - z, x + 3y - z, 2z)$. Is T diagonalizable? If yes, find a basis such that the matrix representation of T with respect to that basis is diagonal. [15%]

(4) Find an orthonormal basis (with the usual inner product in $\mathbf{R}^4$) for the subspace spanned by $\{(1, -1, 0, 1), (0, 1, 0, 2), (1, 0, 1, -2)\}$. [10%]

(5) Let V be a finite dimensional vector space, and $E : V \to V$ be a projection operator, i.e., $E^2 = E$. Prove or disprove: E is diagonalizable. [15%]

(6) Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & 0 \end{pmatrix}.$$

(a) Find the minimal polynomial of A. [8%]
(b) Find the Jordan canonical form of A. [7%]
(c) Find a Jordan basis for A. [15%]

Ten points for each problem. Please write down the detail of your computation.

1. Evaluate $\lim_{n\to\infty}(\tan\frac{1}{n})^{1/n}$.
2. Evaluate $\frac{d}{dx}\int_{\log_2|\sec x|}^{x^x}\sin t^5 dt$.
3. Evaluate $\int_0^\infty \frac{\sin x}{e^{2x}}dx$.
4. Evaluate $\int_{-2}^{2}\int_0^{\sqrt{4-x^2}}(1+xy^2)dydx$.
5. Compute the Taylor series for $\arcsin x$ at $x=0$.
6. Plot the graph of $f(x)=1+\frac{3}{x}-\frac{1}{x^3}$ and indicate all its asymptotes, inflection points and relative maximum and minimum points.
7. Solve the differential equation $xy'(x)=y^3(x)-y'(x)$ with $y(0)=1$.
8. Let a and b be constants, find the parametric equations of the tangent line and normal line to the helix $\mathbf{r}(t)=(a\cos t, a\sin t, bt)$ at time t.
9. Find the maximum and minimum values of $f(x,y)=e^{-xy}$ subject to the constraint $x^2+y^2\leq 1$.
10. Show that the Laplace equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$ in polar coordinates is

$$\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}=0.$$

Entrance Exam of Numerical Analysis for the Master Program of Scientific Computing

Full marks are 100; questions with the marks are indicated.

I. (20) Give the definitions of convergence and stability of numerical methods, address their differences and relations, and provided simple examples to illustrate them.

II. (20) Prove

$$\{\int_a^b wf(x)g(x)dx\}^2 \leq \{\int_a^b wf^2(x)dx\}\{\int_a^b wg^2(x)dx\}, \tag{1}$$

where $w \geq 0$ on $[a, b]$.

III (20) Give the trapezoidal and midpoint rules for the integral, $I = \int_a^b f(x)dx$. Show that when $f''(x) \geq 0$ on $[a, b]$, the approximate integrations by the trapezoidal and midpoint rules are the upper and lower bounds of I, respectively.

IV. (20) Let the linear algebraic equations be $Ax_i = b_i, 1 < i << n$, where $A \in R^{n\times n}$, $x_i \in R^n$ and $b_i \in R^n$. Suppose that matrix A is positive definite and symmetric. Give the Choleski method for solving them, provide the computer storage needed and the order of CPU time, with respect to n.

V. (20) Give the Newton iteration method to evaluate $\sqrt{a}$, $a > 0$, and then use Language C or *Fortran* to write a computer program. In the program, choose 1 as the initial approximation of $\sqrt{a}$, and the termination condition of the iteration that the initial errors of the residuals are reduced by a factor $\frac{1}{2}10^{-6}$.

Answer all 5 problems. Each problem carries 20 points

1. (a) Show that if a function f is differentiable at a point x, then f is continuous at x. (10%)

   (b) Suppose $g: \mathbf{R} \to \mathbf{R}$ has the property that for all $x \in \mathbf{R}$, $-x^2 \le g(x) \le x^2$. Show that g is differentiable at 0 and $g'(0) = 0$. (10%)

2. Let $f_n: [0,1] \to \mathbf{R}$ be defined by $f_n(x) = nxe^{-nx^2}$, for each $n \in \mathbf{N}$.

   (a) Evaluate $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$ and $\int_0^1 \lim f_n(x)\,ds$. (10%)

   (b) Does f_n converge uniformly? Why? (10%)

3. Show that the series

$$\sum_{n=1}^{\infty} \ln(1 + \frac{(-1)^n}{n^p})$$

is divergent when $0 < p \le 1/2$; conditionally convergent when $1/2 < p \le 1$; and absolutely convergent when $p > 1$.

Hint: Show that for large n's,

$$\frac{(-1)^n}{n^p} - \frac{1}{2n^{2p}} \le \ln(1 + \frac{(-1)^n}{n^p}) \le \frac{(-1)^n}{n^p} - \frac{1}{6n^{2p}}.$$

4. (a) Show that a continuous function $f: [a,b] \to \mathbf{R}$ is integrable. (10%)

   (b) Evaluate the line integral

$$\int_C \frac{x\,dy - y\,dx}{x^2 + y^2}$$ (10%)

   where C is a smooth simple closed curve such that the origin is outside C.

5. Define the function $f: \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} \frac{x\sqrt{x^2+y^2}}{|y|} & \text{if } y \ne 0 \\ 0 & \text{if } y = 0 \end{cases}$$

   (a) Show that f is not continuous at $(0,0)$. (6%)

   (b) Show that f has directional derivatives in all directions at $(0,0)$. (7%)

   (c) Show that for all $c \in \mathbf{R}$, there is a vector $\mathbf{v}$ of norm 1 such that the directional derivative along $\mathbf{v}$, $D_{\mathbf{v}} f(0,0)$ is equal to c. (7%)

In the following problems, C = The set of complex numbers, R = The set of real numbers, $D_r(\alpha) = \{z \in \mathbf{C} : |z-\alpha| < r\}$ and $C_r(\alpha) = \{z \in \mathbf{C} : |z-\alpha| = r\}$. Each of the following problem is worth 20 points.

Problem 1. By Riemann mapping theorem, there is an one to one holomorphic mapping that carries any simply connected region $\Omega \neq \mathbf{C}$ onto $D = D_1(0)$. Let $\Omega = \{z \in \mathbf{C} : 0 < \operatorname{Re} z < 1\} \setminus \{x \in \mathbf{R} : 0 < x < \frac{1}{2}\}$. Find explicitly an one to one holomorphic function φ that maps Ω onto D.

Problem 2. Let $\omega = a + bi$, $a, b \in \mathbf{R}$ and $|\omega| < 1$ Given $0 < r < 1$, consider the set

$$B_r(\omega) = \left\{ z \in \mathbf{C} : \frac{|z-\omega|}{|1-\overline{\omega}z|} < r \right\}.$$

Describe $B_r(\omega)$ explicitly. What happens when $|\omega| \to 1$?

Problem 3. Let C be the positive oriented circle $C_8(0)$. Use the residue theorem to compute

$$\int_C \tan z \, dz.$$

Problem 4. Let $f = u + iv$ be an entire function such that $au + bv > c$ for some $a, b, c \in \mathbf{R}$. Show that f must be a constant (Hint: Find a suitable nonconstant entire function g such that $e^{g(f(z))}$ is constant).

Problem 5. (a, 10 points) Suppose that g is analytic on Ω and $\overline{D_r(\alpha)} = D_r(\alpha) \cup C_r(\alpha) \subseteq \Omega$. Prove that

$$g(\alpha) = \frac{1}{\pi r^2} \int_{D_r(\alpha)} g(z) \, dxdy.$$

(b, 10 points) Let f be analytic on $D_2(0) \setminus \{0\}$ such that the improper integral

$$\int_D |f(z)|^2 \, dxdy = \lim_{r \to 0} \int_{D \setminus D_r} |f(z)|^2 \, dxdy$$

exists (where $D = D_1(0)$ and $D_r = D_r(0)$). Use (a) to show that 0 is a removable singularity of f.