國立中山大學 107 學年度碩士暨碩士專班招生考試試題

科目名稱：基礎數學【應數系碩士班甲組】　　　　　題號：424001

※本科目依簡章規定「不可以」使用計算機(問答申論題)　　　　共 1 頁第 1 頁

共十題，每題 10 分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. Use implicit differentiation to find the tangent line at $(2,2)$ of the graph of the function $2x^3 - 3y^2 = 4$. Also find the second derivative $\frac{d^2y}{dx^2}$ at $(2,2)$.

2. Evaluate the limits.

   (a) $\lim_{x\to\frac{\pi}{4}} \tan 2x \cdot \tan(\frac{\pi}{4} - x)$.

   (b) Suppose $\lim_{x\to\infty} f'(x) = A$, $a > 0$, find $\lim_{x\to\infty}\{f(x+a) - f(x)\}$.

3. Consider a segment of the curve described by the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ in the first quadrant (i.e., when $0 \le x \le 1$ and $0 \le y \le 1$).

   (a) Find the length of the curve.

   (b) Find the area of the surface generated by revolving the curve about the x-axis.

4. Find the antiderivative $\int \frac{e^x+1}{e^{2x}-e^x+2}\,dx$.

5. Consider the power series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(n+1)\ln(n+1)}$.

   (a) Determine its radius of convergence.

   (b) Determine its interval of convergence.

6. Find and classify the critical points of the function $f(x,y) = x^4 + y^4 - 4xy$.

7. Find the area of the sphere $x^2+y^2+z^2 = 4$ lying inside the cylinder $(x-1)^2+y^2 = 1$.

8. Diagonalize the matrix $\begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix}$ with the real eigenvalues $\lambda = -2,\ -1,\ 0$.

9. The given set

$$\left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} \right\}$$

   is a basis for a subspace W. Use the Gram-Schmidt process to produce an orthogonal basis for W.

10. Make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $x_1^2 + 10x_1x_2 + x_2^2$ into a quadratic form with no cross-product term. Give P and the new quadratic form.

試題隨卷繳回

國立中山大學107學年度碩士暨碩士專班招生考試試題

科目名稱：微積分【應數系碩士班乙組】　　　　　　　　　　　　題號：424002

※本科目依簡章規定「不可以」使用計算機(問答申論題)　　　　　　共1頁第1頁

計算題：共7題，子題分數平均分配。答題時，每題都必須寫下題號與詳細步驟。

[1]. (10%) Let $F(x) = 2^x + \dfrac{1}{\sqrt{x^2+1}} + \int_0^x \cos(\pi s^2)\,ds$. Find $F'(x)$ and $F''(x)$.

[2]. (16%)

(a) Find the Taylor series of $f(x) = \sin(x)$ at $x = \frac{\pi}{2}$.

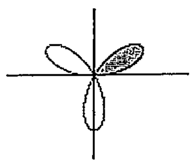
(b) Evaluate $\lim\limits_{x\to 0} \dfrac{x^3 - \sin x^3}{x^9}$

[3]. (20%) Evaluate the following integrals.

(a) $\int_0^\pi e^x \cos x\,dx$　　　(b) $\int_{-\infty}^{\infty} \dfrac{1}{x^2+2x+5}dx$

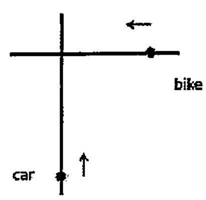
[4]. (14%) Find the interval of convergence for the power series $\sum\limits_{n=1}^{\infty} \dfrac{3^n}{n}(5x-1)^n$.

[5]. (12%) Find the area of one leaf of the three-leaf rose $r = \sin 3\theta$.



[6]. (14%) Evaluate the iterated integral $\int_0^{\frac{\sqrt{\pi}}{2}} \int_y^{\frac{\sqrt{\pi}}{2}} \sin(x^2)\,dxdy$.

[7]. (14%) A bike is traveling west at 20 km/h and a car is traveling north at 80 km/h. Both are headed for the intersection of the two roads. At what rate are the bike and the car approaching each other when bike is 30 m and car is 40 m from the intersection?



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試題隨卷繳回

國立中山大學107學年度碩士暨碩士專班招生考試試題

科目名稱：線性代數【應數系碩士班丙組】 題號：424003

※本科目依簡章規定「不可以」使用計算機(問答申論題) 共1頁第1頁

請詳列計算証明之過程. 作答時請標明題號. 共七題.

1. Find the general solution of $\begin{cases} x_1 - x_2 + x_3 + 2x_4 = 2 \\ x_1 + x_2 - 2x_3 - x_4 = 1 \end{cases}$. (10%)

2. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. Is A invertible? Find A^{-1} if it exists. (10%)

3. Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$. Find a basis for the null space of A and a basis for the image space of A. (15%).

4. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Find an orthogonal P such that $P^{-1}AP$ is diagonal. Also, find a matrix B such that $B^{107} = A$. (15%)

5. Let A be an $n \times n$ matrix and $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a subset of $\mathbf{R}^n$.
   (a) Prove or disprove: If A is invertible and $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is independent, then $\{A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_k\}$ is independent. (10%)
   (b) Prove or disprove: If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ and $\{A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_k\}$ are independent, then A is invertible. (10%)

6. Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x, y) = (x - 2y, x + y)$. Find the matrix of T relative to the ordered basis $B = \{(1, 1)^T, (-1, 1)^T\}$. (15%)

7. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$. Find $A^{14} + A^{13} - 5A^{12} + A^5 + A^4 - 5A^3 + A^2 - A + 2I$. (15%)

試題隨卷繳回

國立中山大學107學年度碩士暨碩士專班招生考試試題

科目名稱：高等微積分【應數系碩士班丙組】 題號：424004
※本科目依簡章規定「不可以」使用計算機(問答申論題) 共1頁第1頁

1. [10%] Find the Taylor series expansion for the function

$$f(x) = \frac{1}{1+2x^2}$$

about $x = 0$ and find the convergence of interval of the series.

2. [15%] Define $f: \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x,y) = \frac{y^{5/2}}{x^2+y^2}$$

for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Is f continuous at $(0,0)$? Verify your assertion.

3. [15%] Let $\{a_n\}$ be a positive sequence with $\sum a_n$ divergent. Show that the series

$$\sum \frac{a_n}{1+a_n}$$

also diverges.

4. [15%] Let $\{f_n\}$ be a sequence of continuous functions defined on $[0,1]$, and suppose that the limit $\lim_{n\to\infty} f_n(x) = f(x)$ exists for any $x \in [0,1]$.
(1)[7%] Is f continuous on $[0,1]$? Verify your assertion.
(2)[8%] Is it true that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx?$$

Verify your assertion.

5. [15%] Show that the equation

$$x^2 + x + y + \sin(x^2+y^2) = 0$$

determines a unique solution y as a function x near the point $(0,0)$ and show that this unique solution is differentiable at 0. Find the derivative $y'(0)$.

6. [15%] Show that for any continuous function $f: [0,1] \to [0,1]$, there exists a point $\xi \in [0,1]$ for which $f(\xi) = \xi$.

7. (1)[8%] Is the intersection

$$\bigcap_{k=1}^{n} V_k$$

of open sets $V_1, \ldots, V_n$ in some metric space X open in X? Verify your assertion.
(2)[7%] Is your assertion in (1) still true if the finite intersection is replaced with a countable intersection of open sets in X? Verify your assertion.

試題隨卷繳回

國立中山大學107學年度碩士暨碩士專班招生考試試題

科目名稱：線性代數【應數系碩士班乙組】 題號：424005
※本科目依簡章規定「不可以」使用計算機(問答申論題) 共1頁第1頁

計算題：共6題，子題分數平均分配。答題時，每題都必須寫下題號與詳細步驟。

[1]. (18%) Answer the following questions.
(a) What is the vector space?
(b) What is the linear transformation?
(c) What is the rank of a matrix?

[2]. (14%) Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & -1 & 2 & -2 \\ 0 & 0 & 1 & 3 \\ 3 & 4 & 2 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

(a) Solve the linear system $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.
(b) Compute $\det(A)$.

[3]. (18%) Let

$$A = \begin{bmatrix} 2 & -6 & 3 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -6 & 3 & 3 \end{bmatrix}.$$

(a) Find the characteristic polynomial.
(b) Find the eigenvalues and the corresponding eigenvectors.
(c) Find a matrix C and a diagonal matrix D such that $D = C^{-1}AC$.

[4]. (16%) Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix}.$$

(a) Use the Gram-Schmidt process to find an orthonormal basis for the column space of A.
(b) Solve the least squares problem $A\mathbf{x} = \mathbf{b}$.

[5]. (16%) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by

$$T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 + 3x_3 \\ x_1 - x_2 + 2x_3 \\ 3x_1 + 2x_2 \end{bmatrix}.$$

(a) Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x} = (x_1, x_2, x_3)^T$ in $\mathbb{R}^3$.
(b) Let $\beta = ([1,1,1], [1,1,0], [1,0,0])$ be an ordered basis of $\mathbb{R}^3$.
Find the matrix representation $B = [T]_\beta$ of T with respect to β.

[6]. (18%) Let A be an $m \times n$ matrix. Show that
(a) The nullspace of A^TA is the nullspace of A.
(b) A^TA and A have the same rank.

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試題隨卷繳回

國立中山大學 107 學年度碩士暨碩士專班招生考試試題

科目名稱：機率與統計【應數系碩士班甲組】　　　　題號：424006

※本科目依簡章規定「不可以」使用計算機(問答申論題)　　　　共1頁第1頁

答題時，每題須寫下題號與詳細步驟。請依題號順序作答，不會作答題目請寫下題號並留空白。

Notation:
i.i.d.: identically independently distributed; pdf: probability density function; MLE: maximum likelihood estimator; $exp(\theta)$ random variable means a random variable with exponential distribution with a parameter θ and its pdf is $f(x|\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}, x > 0$. $\bar{X}$ is the sample mean of $X_1, \ldots, X_n$. $Bin(n,p)$ indicates the Binomial distribution with n independent Bernoulli trail and each trail has success rate p. Unif[a,b] represents the uniform distribution within [a,b].

1. (15%) The joint moment generating function for random variables U and V is defined as M(s,t)=$E(\exp(sU+tV))$. X and Y are independent random variables with common moment generating function $M(t) = \exp(5t^2)$. Let $U = X + Y + 3$ and $V = 2X - 2Y$. What is the joint moment generating function for U and V?

2. (15%) Assume the distribution of N is $Bin(m,p)$. Conditional on $N = n$, the distribution of Y is $Bin(n,q)$. What is the unconditional distribution of Y?

3. (15%) Let $X_1, \ldots, X_{2n}$ be iid Unif[0,3]. The order statistics are $X_{(1)} < X_{(2)} < \cdot < X_{(2n)}$ What is the expectation of $X_{(n)}$?

4. (15%) Let $Y_i \sim Bin(n_i, p_i), i = 1, \ldots, m$, be mutually independent. Please derive the likelihood ratio test with significance level α for the null hypothesis

$$H_0 : p_1 = \ldots = p_m$$

against the alternative hypothesis that not all the p_i are equal. You have to specify the test statistic and the asymptotic rejection region.

5. (20%) $X_1, \ldots, X_n$ are i.i.d $\exp(\theta)$. Please answer the following questions.
(a) (5%) Prove $\bar{X}$ and $\frac{X_1}{\bar{X}}$ are independent.
(b) (15%) Use (a) to derive the UMVUE for the parameter $P(X_1 > t) = e^{-t/\theta}$.

6. (20%) $X_1, \ldots, X_n$ are independent random variables with X_i being distributed with $N(\mu, w_i\sigma^2)$, where w_i are known constants and μ and σ^2 are unknown parameters).
(a) (10%) Please find the MLE for μ.
(b) (10%) Calculate and the mean squared errors of the MLE derived from (a) and $\bar{X}$ for μ. Which one has smaller mean squared error (you have to prove your answer)?

試題隨卷繳回