國立中山大學 105 學年度碩士暨碩士專班招生考試試題

科目名稱：機率與統計【應數系碩士班甲組】　　　　題號：424006
※本科目依簡章規定「不可以」使用計算機(問答申論題)　　　　共1頁第1頁

答題時，每題須寫下題號與詳細步驟。請依題號順序作答，不會作答題目請寫下題號並留空白。

Notation: cdf: cumulative distribution function; pdf: probability density function; mle: maximum likelihood estimator; iid: identically independently distributed. z_α is the cut-off point of standard normal distribution with the upper tail probability α. $\chi^2_{\alpha,n}$ is the cut-off point of chi-square distribution with degrees of freedom n and upper tail probability α. Use significance level α for all the tests below.

1. (15%)An urn contains five red and two white balls. A ball is drawn, and then it and another ball of the same color are placed back in the urn. Finally, a second ball is drawn. Given the second ball drawn is red, what is the probability that the first ball drawn was also red?

2. (15%) A point is randomly chosen in the interior of an ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let X and Y be the x and y coordinate of the point, respectively. Given $X=s$, find the conditional density of $Y=t$.

3. Suppose that $X_1,\ldots,X_n$ are iid with cdf
$$F(t)=1-e^{-(3x-\theta)}, x\geq \frac{1}{3}\theta,$$
and $F(t)=0$ otherwise. Here θ is the unknown parameter and $\theta>0$.
(a)(10%)Please find the mle of the parameter θ.
(b)(10%)Find a one dimensional sufficient statistic for θ.

4. (15%)Let $X_1,\ldots,X_n$ be iid random variables from an exponential distribution with density $f(x)=\lambda e^{-\lambda x}, x>0, \lambda>0$. Find the uniformly most powerful test for $H_0:\lambda=1$ against $H_a:\lambda<1$ (significance level $\alpha=0.05$). Note that you have to specify the test statistic and the rejection region (significance level is α).

5. Let $X_1,\ldots,X_n$ be iid normal random variables with mean 4 and unknown variance σ^2. Please answer the following questions. Significance level is α.
(a)(10%) Please test the $H_0:\sigma=3$ against $H_a:\sigma<3$. Note that you have to specify the test statistic and the rejection region (significance level is α).
(b)(10%) Find the $1-\alpha$ confidence interval for σ.

6. Let $X_1,\ldots,X_n$ be independent random variables with $Var(X_i)=i^2, i=1,\ldots,n$ and common mean μ. Let $Y=\sum_{i=1}^n d_iX_i$, where d_i are scalars. Please answer the following questions.
(a) (5%)Find a condition on the $d_i, i=1,\ldots,n$ so that Y is an unbiased estimator of μ.
(b) (10%)Given Y is unbiased for μ, determine what values of $d_i, i=1,\ldots,n$ would make Y have minimum variance? You have to prove your result.

試題隨卷繳回

國立中山大學105學年度碩士暨碩士專班招生考試試題

科目名稱：基礎數學【應數系碩士班甲組】 題號：424001

※本科目依簡章規定「不可以」使用計算機(問答申論題) 共1頁第1頁

共十題，每題 10 分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. 設首項爲 $a_1 = 1, b_1 = 1$ 的數列 $\langle a_n \rangle$、$\langle b_n \rangle$，且對於所有的自然數 n，$\begin{cases} a_{n+1} = a_n - 2b_n \\ b_{n+1} = a_n + 4b_n \end{cases}$ 恆成立，求 a_n。

2. 設矩陣 $A = \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix}$, $n \in \mathbb{N}$，求出 $A^{n+2} - 2A^{n+1} + A^n$。

3. 設 $a, b, c, x, y, z \in \mathbb{R}$，且 $a^2 + b^2 + c^2 = 16$, $x^2 + y^2 + z^2 = 25$，求 $\begin{vmatrix} 1 & 2 & 2 \\ a & b & c \\ x & y & z \end{vmatrix}$ 之絕對值的最大值。

4. 設 a, b, c 爲方程式 $2x^3 + 4x^2 + 6x - 1 = 0$ 的三個根，求行列式

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

的值。

5. 若 $\lim_{x \to 1} \frac{x^5 + ax + b}{(x-1)^2}$ 的極限存在，求 a, b。

6. 設 $f(x) = \frac{(x-1)(x-2)(x-3)(x-4)}{4}$，求導函數 $f'(1)$。

7. 設 $F(x) = \int_0^{x^2} \frac{1}{1+\sin^2 t}\, dt$，則導函數 $F'(x)$ 爲何？

8. 設 $x^3 - 6x^2 - 15x - k = 0$ 有三相異實根，求 k 的範圍。

9. 求 $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx\, dy}{(1+x^2+y^2)^2}$ 的值。

10. 求定積分 $\int_0^{\pi} \ln(\sin x)\, dx$ 的值。

試題隨卷繳回

國立中山大學105學年度碩士暨碩士專班招生考試試題

科目名稱：線性代數【應數系碩士班乙組】　　題號：424005

※本科目依簡章規定「不可以」使用計算機(問答申論題)　　共1頁第1頁

共八道題。答題時，每題須寫下題號與詳細步驟。 請依題號順序作答，不會作答題目請寫下題號並留空白。

1. [10%] Find a 3×3 matrix P satisfying

$$P\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. [10%] Find the Jordan form of the matrix

$$\begin{pmatrix} 6 & -2 & -1 & -1 & 0 \\ -2 & 6 & 1 & 1 & 0 \\ 0 & 0 & 6 & 2 & -1 \\ 0 & 0 & 2 & 6 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

3. [10%] Let V be a vector space, and let $T : V \to V$ be a linear transformation. Prove that $T^2 = 0$ if and only if $\text{range}(T) \subseteq \ker(T)$. Give an example of T for which $\text{range}(T) = \ker(T)$.

4. [10%] Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. Show that either $T(v) = 0$ for some non-zero vector $v \in V$ or $T(x) = u$ has a solution x in V for every $u \in V$.

5. [15%] Let A be an $m \times n$ real matrix whose rank is n. Show that A^tA is invertible.

6. [15%] Let A be an $n \times n$ matrix with $A^k = 0$ for some $k \in \mathbb{N}$. Show that $I_n - A$ is invertible.

7. [15%] Let V be a vector space with $\dim V = n$. Suppose that $T : V \to V$ is a linear transform and that there exists some $v \in V$ satisfying $A^{n-1}v \neq 0$ and $A^n v = 0$. Show that A admits the following matrix representation

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

with respect to some basis of V.

8. [15%] Let A and B be $n \times n$ real symmetry matrices, and A be positive semidefinite. Show that there exists an invertible matrix P so that P^tAP and P^tBP are diagonal matrices.

The end of the paper

試題隨卷繳回

國立中山大學 105 學年度碩士暨碩士專班招生考試試題

科目名稱：微積分【應數系碩士班乙組】 題號：424002
※本科目依簡章規定「不可以」使用計算機(問答申論題) 共1頁第1頁

計算題：共7題，子題分數平均分配。答題時，每題都必須寫下題號與詳細步驟。

[1]. (16%) Evaluate the following limits.

(a) $\lim_{x\to\infty} \frac{\int_1^{x^2} \ln\left(e^{3t-2}\right) dt}{x^4}$

(b) $\lim_{n\to\infty} \left[\frac{1}{\sqrt{4n^2-1^2}} + \frac{1}{\sqrt{4n^2-2^2}} + \frac{1}{\sqrt{4n^2-3^2}} + \cdots + \frac{1}{\sqrt{4n^2-n^2}}\right]$

[2]. (14%) Solve the differential equation $x^3 y'(x) = 2e^{1/x^2} - 2y(x)$ with $y(1) = -e$.

[3]. (14%) Let $f(x) = \int_{-x^2}^{x^2} e^{-\pi t^2} dt$. Find $f'(x)$ and $f''(x)$.

[4]. (12%) Find the Taylor series of $\tan^{-1}(x)$ at $x = 0$.

[5]. (20%) Evaluate the following integrals.

(a) $\int \frac{1}{2-\cos x} dx$ (b) $\int_0^3 \left|x^2 - 3x + 2\right| dx$

[6]. (12%) Let $f(x, y) = \sqrt{6 - 2x^2 - 2y^2}$ and $g(r, t, \theta) = \left(r\sqrt{t}\cos\theta, r\sqrt{t}\sin\theta\right)$.
Use the chain rule to compute the gradient $\nabla f(g(r, t, \theta))$.

[7]. (12%) Find the relative maximum and relative minimum for $g(x) = 5x^3 - 3x^5$.

======== 全卷完 ========

試題隨卷繳回

國立中山大學105學年度碩士暨碩士專班招生考試試題

科目名稱：高等微積分【應數系碩士班丙組】　　題號：424004
※本科目依簡章規定「不可以」使用計算機(問答申論題)　　共1頁第1頁

答題時，每題須寫下題號與詳細步驟。 請依題號順序作答，不會作答題目請寫下題號並留空白。

1.(25 points) Suppose that f is a mapping from a complete metric space S into S such that

$$d(f(x), f(y)) \leq d(x, y)$$

for all $x, y \in S$, where $d(x, y)$ denote the distance from x to y. Show that there is a unique x_0 such that $f(x_0) = x_0$.

2.(25 points) Let G be an open, simply-connected domain in $\mathbb{R}^2$ with smooth boundary ∂G. If $P = P(x_1, x_2)$ and $Q = Q(x_1, x_2)$ are smooth in $\mathbb{R}^2$, then show that

$$\int_G \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \right) dx_1 dx_2 = \oint_{\partial G} P dx_1 + Q dx_2$$

where the line integral is taken into a counter-clockwise direction.

3.(25 points) Let m, n be positive integers. Show that

$$\sum_{0<m<\infty, 0<n<\infty} \frac{(-1)^{m+n}}{m^3 + n^3}$$

exists.

4.(25 points) Given the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = \begin{cases} \frac{x_2^3}{x_1^2 + x_2^2}, & (x_1, x_2) \neq (0, 0) \\ 0, & x_1 = x_2 = 0 \end{cases}$$

Show that f is continuous at $(0, 0)$.

試題隨卷繳回

國立中山大學105學年度碩士暨碩士專班招生考試試題

科目名稱：線性代數【應數系碩士班丙組】 題號：424003

※本科目依簡章規定「不可以」使用計算機(問答申論題) 共1頁第1頁

Please write down all the detail of your computation and solution.

1. (10%) Find the third column of the following product matrix

$$\begin{bmatrix} \pi & \sqrt{e} & \frac{1}{3} & \sqrt{2} \\ 3.7 & 10^5 & 7 & 0 \\ \ln 2 & i & \sin 3 & -1 \end{bmatrix} \begin{bmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ 0.2 & -0.3 & 0.1 \\ 2 & -1 & -1 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} -1 & 2.3 & 1 & -3 \\ \sqrt{5} & \frac{1}{2} & 1 & 2 \\ 3 & -2 & 1 & \sqrt{2} \end{bmatrix}.$$

2. (15%) Find permutation matrix P, lower triangular matrix L and upper triangular matrix U such that

$$\begin{pmatrix} 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} = PLU.$$

3. (15%) Let $m \times n$ matrix A be the matrix representation of linear transformation L. Show that (1) L is one-to-one $\iff$ all columns of A are linearly independent, (2) L is onto $\iff$ all rows of A are linearly independent.

4. (15%) For every $n \times n$ complex matrix $A = (a_{ij})_{n \times n}$, define

$$||A|| = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}.$$

Show that it is a matrix norm compatible with the vector norm

$$||(x_1, x_2, \cdots, x_n)^T|| = \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

5. (15%) Let

$$A = \begin{pmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{pmatrix}.$$

Find and prove the necessary and sufficient condition on a and b such that A is symmetric positive definite.

6. (15%) Let A be an diagonalizable matrix and t be a parameter. Consider the linear equation $(A - tI)\mathrm{x} = \mathrm{b}$. (1) Discuss the existence and uniqueness of x in terms of the values of t and b, eigenvalues and eigenvectors of A. (2) Find all of its solutions if they exist.

7. (15%) Let $n \times n$ matrix A have all entries 1. Find all of its eigevalues, corresponding eigenvectors, and its Jordan canonical form.

試題隨卷繳回