國立中山大學102學年度碩士暨碩士專班招生考試試題

科目名稱：基礎數學【應數系碩士班甲組】　　　　題號：424001
※本科目依簡章規定「不可以」使用計算機　　　　共1頁第1頁

答題時，每題須寫下題號與詳細步驟。請依題號順序作答，不會作答題目請寫下題號並留空白。

1. (10%) Evaluate $\lim_{x \to 0} 2^{x\cos(1/x)}$.

2. (10%) Evaluate $\int_0^{\pi/2} \frac{\cot x}{1+\csc x}\,dx$.

3. (10%) Let $1 < a < b < \infty$. Prove that $\arctan b - \arctan a \le (b-a)/2$.

4. (10%) Calculate the area of the region Ω enclosed by the curves

$$x^3 + y^3 + 3x^2y + 3xy^2 - 3x - y = 0 \quad \text{and} \quad x^2 + y^2 + 2xy - x + y = 0.$$

5. (20%) Let $f(x) = 2 + 3x + x^2 + 2x^3 + 3x^4 + x^5 + 2x^6 + 3x^7 + x^8 + \cdots$. Find the interval of convergence for $f'(x)$.

6. (20%) Determine all possible values of a such that the matrix $\begin{pmatrix} 6 & -2 & 0 \\ -2 & a & -3 \\ 0 & -3 & 4 \end{pmatrix}$ is positive definite.

7. (20%) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$. Find matrices D and Q such that $A = QDQ^{-1}$, where D is the Jordan normal form of A and Q^{-1} is the inverse of Q.

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國立中山大學102學年度碩士暨碩士專班招生考試試題

科目名稱：數理統計【應數系碩士班甲組】　　題號：424005
※本科目依簡章規定「不可以」使用計算機　　共2頁 第1頁

共十題，每題 10 分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. If the moment-generating function of X is $M(t) = \exp(166t + 200t^2)$, find
   (a) (5 分) The variance of X.
   (b) (5 分) $P(148 \le X \le 172)$.

2. Let X and Y have a bivariate normal distribution with parameters $\mu_X = 2.8$, $\mu_Y = 110$, $\sigma_X^2 = 0.16$, $\sigma_Y^2 = 100$, and $\rho = 0.6$. Compute
   (a) (5 分) $P(106 < Y < 124)$.
   (b) (5 分) $P(106 < Y < 124 | X = 3.2)$.

3. Let Z_1, Z_2, and Z_3 have independent standard normal distributions, $N(0, 1)$. Find the distribution of
$$W = \frac{Z_1}{\sqrt{(Z_2^2 + Z_3^2)/2}}.$$

4. Approximate $P\big(39.75 \le \overline{X} \le 41.25\big)$, where $\overline{X}$ is the mean of a random sample of size 32 from a distribution with mean $\mu = 40$ and variance $\sigma^2 = 8$.

5. Suppose that we have a random sample $X_1, X_2, \ldots, X_n$ from a population that is $N(\mu, \sigma^2)$. We plan to use to estimate $\hat{\Theta} = \sum_{i=1}^{n}(X_i - \overline{X})^2/c$. Compute the bias in $\hat{\Theta}$ as an estimator of σ^2 as a function of the constant c.

6. Let $f(x) = (1/\theta)x^{(1-\theta)/\theta}$, $0 < \theta < \infty$, and $0 < x < 1$. Find the maximum likelihood estimator for θ. Is it an unbiased estimator for θ?

7. Let X be a random variable with mean μ and variance σ^2. Given two independent random samples of sizes n_1, and n_2, with sample means $\overline{X}_1$ and $\overline{X}_2$, show that $\overline{X} = a\overline{X}_1 + (1-a)\overline{X}_2$, $0 < a < 1$ is an unbiased estimator for μ. If $\overline{X}_1$ and $\overline{X}_2$ are independent, find the value of a that minimizes the standard error of $\overline{X}$.

8. Consider the probability density function
$$f(x) = c(1 + \theta x), \quad -1 \le x \le 1$$
   (a) (5 分) Find the value of the constant c.
   (b) (5 分) What is the moment estimator for θ?

9. Let $X_1, \ldots, X_n$ be iid Poisson(λ). Find a uniformly most powerful (UMP) test of $H_0 : \lambda \le \lambda_0$ versus $H_1 : \lambda > \lambda_0$.

10. The fraction of defective integrated circuits produced in a photolithography process is being studied. A random sample of 300 circuits is tested, revealing 13 defectives. Calculate a 95% two-sided confidence interval on the fraction of defective circuits produced by this particular tool.

Cumulative Standard Normal Distribution $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	z
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	0.0
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	0.1
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	0.2
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	0.3
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	0.4
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	0.5
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	0.6
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	0.7
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	0.8
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	0.9
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	1.0
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	1.1
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	1.2
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	1.3
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1.4
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1.5
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1.6
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1.7
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1.8
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1.9
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	2.0
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	2.1
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	2.2
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	2.3
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	2.4
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	2.5
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	2.6
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	2.7
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	2.8
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	2.9
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990	3.0
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993	3.1
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	3.2
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	3.3
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	3.4
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	3.5
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.6
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.7
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	3.8
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	3.9
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	z

(1) A random number N of dice is thrown. Let A_i be the event that $N = i$, and assume that $P(A_i) = 2^{-i}, i \geq 1$. The sum of the score is S.

(a) Find the conditional probability that $N = 2$ given $S = 4$. (10pts)

(b) Find the conditional probability that $S = 4$ given N is even. (10pts)

(2) Let X_1, X_2, X_3 be independent random variables taking values in the positive integers and having mass functions given by $P(X_i = x) = (1 - p_i)p_i^{x-1}$ for $x = 1, 2, \ldots,$ and $i = 1, 2, 3$. Show that

$$P(X_1 < X_2 < X_3) = \frac{(1-p_1)(1-p_2)p_2p_3^2}{(1-p_2p_3)(1-p_1p_2p_3)}.(10pts)$$

(3) Suppose X and Y are independent r.v.'s, with $X \sim \text{Gamma}(\alpha_1, \lambda)$, and $Y \sim \text{Gamma}(\alpha_2, \lambda)$. Find $E(X \mid Z)$, where $Z = X + Y$.(10pts)

(4) Let X and Y be independent random variables each having the uniform distribution on $[0, 1]$. Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$.

(a) Find $E(U)$. (10pts)

(b) Find $cov(U, V)$. (10pts)

(5) Let X, Y, Z be independent and exponential random variables with respective parameters λ, μ, ν. Find $P(X < Y < Z)$.(10 pts)

(6) Let X and Y have the bivariate normal density function

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp\{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\}.$$

(a) Show that X and $Z = (Y - \rho X)/\sqrt{1-\rho^2}$ are independent $N(0, 1)$ variables. (10pts)

(b) Show that $P(X > 0, Y > 0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho$. (10pts)

(7) Let X have the binomial distribution with parameters n and p, and show that

$$E(\frac{1}{1+X}) = \frac{1-(1-p)^{n+1}}{(n+1)p},$$

and find the limit of this expression as $n \to \infty$ and $p \to 0$.(10pts)

Do all the following problems. Show details of your work.

1 Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Find determinant of A and A^{-1} if it exists. (10%)

2. (a) Define $T : \mathbf{R}^4 \to \mathbf{R}^3$ by $T(s,t,x,y) = (s+x+y, s+t-x-y, 3s+t-y)$. Find bases of the kernel of T and the image space of T. (15%)

   (b) Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x,y) = (x-2y, x+y)$. Find the matrix representation of T relative to the ordered basis $B = \{(1,1), (-1,2)\}$ of $\mathbf{R}^2$. (10%)

3. Let A be an $n \times n$ matrix. Prove or disprove: If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_n\}$ are independent sets, then A is invertible. (10%)

4. Let T be a linear transformation on $\mathbf{R}^n$ such that $\text{rank}(T^2) = \text{rank}(T)$. Find $\text{Ker}(T) \cap \text{Im}(T)$. Justify your answer. (15%)

5. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$.

   (a) Find the characteristic polynomial of A. (5%)
   (b) Find the minimal polynomial of A. (5%)
   (c) Find a Jordan form for A. (5%)
   (d) Find a matrix P such that $P^{-1}AP$ is the Jordan form in (c). (10%)

6. Let $< \mathbf{u}, \mathbf{v} > = x_1y_1 + 4x_1y_2 + 4x_2y_1 - x_2y_2$, where $\mathbf{u} = (x_1, x_2)^T$ and $\mathbf{v} = (y_1, y_2)^T$.

   (a) Find a matrix A such that $< \mathbf{u}, \mathbf{v} > = (A\mathbf{u})^T\mathbf{v}$. (5%)
   (b) Does $< \cdot, \cdot >$ define an inner product on $\mathbf{R}^2$? Justify your answer. (10%)

End of Paper

國立中山大學 102 學年度碩士暨碩士專班招生考試試題

科目名稱：微積分【應數系碩士班乙組】　　　　　　　　題號：424004
※本科目依簡章規定「不可以」使用計算機　　　　　　　　共1頁第1頁

計算題：共7題，子題分數平均分配。答題時，每題都必須寫下題號與詳細步驟。

[1]. (18%) Evaluate the following limits.

(a) $\lim_{x\to\infty}\left(\frac{1-4x}{3-4x}\right)^{2-4x}$

(b) $\lim_{n\to\infty}\sum_{k=1}^{n}\cos^2\left(\frac{k\pi}{2n}\right)\frac{\pi}{n}$

[2]. (12%) Solve the differential equation $y'+y=e^x, \quad y(0)=2$.

[3]. (20%) Evaluate the following integrals

(a) $\int_{\sqrt{3}}^{\sqrt{6}}\frac{2}{x\sqrt{x^4-9}}dx$

(b) $\int_{-1}^{0}\frac{2x-2}{x^2+2x+2}dx$

[4]. (10%) Let $f(x,y)=e^{-(x^2+y^2)}$. Find the equation of the tangent plane at the point $(0,1)$.

[5]. (10%) Find the length of the curve $r=1+\cos\theta$ for $0\le\theta\le\pi$.

[6]. (14%) Evaluate the following iterated integral.

$$\int_0^2\int_0^{4-x^2}\frac{xe^{2y}}{4-y}dydx$$

[7]. (16%) Sketch the graph of $f(x)=\sqrt[3]{x^2(9-x)}$.
Determine the open interval on which the graph is increasing, decreasing, concave upward, or concave downward.

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國立中山大學 102 學年度碩士暨碩士專班招生考試試題

科目名稱：數值分析【應數系碩士班乙組】　　　　題號：424006
※本科目依簡章規定「不可以」使用計算機　　　　共1頁第1頁

Please write down all the detail of your computation and answers.

1. (20%) Find (1) the Hermite polynomial, (2) natural (free) cubic spline that interpolates $\sqrt{x}$ at $x_0 = 1$ and $x_1 = 4$.

2. (20%) (1) Write down the algorithm of Newton method to solve a root γ of the nonlinear equation $f(x) = 0$.

   (2) If γ is a simple root of $f(x)$ and the initial value is sufficiently close to γ, show that this iteration converges to γ quadratically by the fixed point theorem.

   (3) What happens if γ is a multiple root? Why?

3. (20%) Derive the composite trapezoidal rule for numerical integration with error formula.

4. (20%) (1) State the pivoting strategies of Gaussian elimination with maximal column pivoting (partial pivoting), scaled column pivoting (implicit scaling) and maximal pivoting (complete pivoting) for $n \times n$ matrix A.

   (2) Which one has the highest accuracy to solve linear systems?

   (3) Which one uses the least CPU time?

5. (20%) Let A be an $n \times n$ nonsingular matrix and B be an $n \times m$ matrix. State the fastest numerical method to compute (1) determinant of A, (2) $A^{-1}B$. How many arithmetic operations are needed in (1), and what is the minimal memory needed in (2)?

國立中山大學 102 學年度碩士暨碩士專班招生考試試題

科目名稱：高等微積分【應數系碩士班丙組】
※本科目依簡章規定「不可以」使用計算機

題號：424003

Question 1 (10 marks)
Evaluate the limit:

$$\lim_{n\to\infty} \frac{\pi + \sqrt{\pi} + \cdots + \sqrt[n]{\pi}}{n}.$$

[You need to give convincing reasons when evaluating this limit.]

Question 2 (10 marks)
Prove that the series

$$\sum_{n=1}^{\infty} \sin(n\pi + \frac{1}{\ln(n+1)})$$

is conditionally convergent.

Question 3 (15 marks)
Prove that the sequence of functions

$$f_n(x) = x^n(1-\cos 2\pi x), \ \ n \ge 1$$

is uniformly convergent over the interval $[0,1]$.

Question 4 (15 marks)
Prove that the series

$$\sum_{n=1}^{\infty} xe^{-nx}$$

is uniformly convergent for $x \ge \delta$ for each fixed $\delta > 0$, but fails to be uniformly convergent for $x > 0$.

Question 5 (12 marks)
Use the Implicit Function Theorem to prove that the equation

$$x^2 + y + \cos(x + \frac{3}{2}\pi e^y) = 0$$

uniquely defines y as a function of x near the point $(0,0)$. Moreover, find the derivative $\frac{dy}{dx}$ at $x = 0$.

Question 6 (14 marks)
Prove that the directional derivative of the function

$$f(x,y) = \frac{x^2 y}{x^4 + y^2} \quad \text{if } (x,y) \ne (0,0) \quad \text{and} \quad f(0,0) = 0$$

exists at $(0,0)$ in every direction, but f is not continuous at $(0,0)$.

Question 7 (12 marks)

Evaluate the surface integral

$$\iint_{\Sigma}(xy+yz+zx)dS$$

where Σ is the portion of the cone $z=\sqrt{x^2+y^2}$ cut by the cylinder $x^2+y^2=2x$.

Question 8 (12 marks)

Evaluate the triple integral

$$\iiint_{\Omega} xyz\,dxdydz$$

where Ω is the region enclosed by the unit sphere $x^2+y^2+z^2=1$ and the coordinate planes $x=0, y=0, z=0$.

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