國立中山大學101學年度碩士暨碩士專班招生考試試題

科目：基礎數學【應數系碩士班甲組】

題號：4048
共1頁 第1頁

答題時，每題須寫下題號與詳細步驟。請依題號順序作答，不會作答題目請寫下題號並留空白。

題目紙上的答案不予計分。

(15%)1. Find all the positive values of p for which the following series converges (You have to give the reasoning).

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^p}$$

(10%)2. What is the nullspace for the following matrix?

$$\begin{pmatrix} 1 & 1 & 0 & -3 \\ 2 & 2 & 1 & -10 \\ 0 & 0 & 1 & -4 \\ 4 & 4 & 3 & -24 \end{pmatrix}$$

(15%)3. $f(x)$ is defined in $(-\pi, \pi)$ as follows:

$$f(x) = \begin{cases} x^{1.2}\sin(1/x) + |x^3 - x| & \text{if } x \in (-\pi, 0) \text{ or } x \in (0, \pi); \\ 0 & \text{if } x = 0. \end{cases}$$

Find all the values of x at which f is differentiable. (You have to give the reasoning).

4. Define matrix A as follows:

$$\begin{pmatrix} 3 & \sqrt{6} \\ \sqrt{6} & 4 \end{pmatrix}$$

Answer the following two questions:

(15%)(a) Assume A can be expressed as KDK^t, where D is a diagonal matrix and K^t is the transpose of matrix K. Find K and D.

(15%)(b) Calculate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp^{-(3x^2+2\sqrt{6}xy+4y^2)}\, dxdy$.

(10%)5. $f(x, y)$ is defined as follows:

$$f(x, y) = \int_{x^2+1}^{2} \frac{1}{1 + y^2t^4} dt.$$

Calculate $\frac{\partial f}{\partial x}$.

(20%)6. Find the maximum of $f(x, y) = e^{xy}$ when x and y are both positive and constrained by the equation $x^2 + y^2 = 8$.

國立中山大學 101 學年度碩士暨碩士專班招生考試試題

科目：線性代數【應數系碩士班乙組、丙組】

題號：4049
共 1 頁 第 1 頁

1. (20 points) Let $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined as the reflection of the plane through the line $3x=5y$. Represent T as a matrix when we consider the domain of T equipped with the ordered basis B_1 and the range space of T equipped with the ordered basis B_2, where

$$B_1=\{\begin{pmatrix}3\\-5\end{pmatrix},\begin{pmatrix}5\\3\end{pmatrix}\} \quad \text{and} \quad B_2=\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\}.$$

2. (20 points) Express the quadratic form

$$f(x,y,z)=x^2+y^2+z^2-2xy+2xz-2yz$$

as

$$\begin{pmatrix}x & y & z\end{pmatrix}A\begin{pmatrix}x\\y\\z\end{pmatrix}$$

for some suitable symmetric 3×3 matrix A. By diagonalizing A, find the maximum and minimum values of f on the unit sphere in $\mathbb{R}^3$, and find all points on the unit sphere where the extrema are assumed.

3. (20 points) Find an invertible matrix S such that $D=S^{-1}AS$ is a diagonalization of the matrix

$$A=\begin{bmatrix}3/4 & 1/4 & 0\\ 1/4 & 3/4 & 0\\ -1/4 & -1/4 & 1/2\end{bmatrix}.$$

Find $\lim_{n\to\infty} A^n$.

4. (20 points) Find a Jordan canonical form and a Jordan basis for the given matrix

$$A=\begin{bmatrix}3 & 6 & 0 & 0 & 0\\ 0 & 3 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & -1\\ 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -1\end{bmatrix}.$$

5. (a) (5 points) Prove that the product of all eigenvalues (repeated according to algebraic multiplicities) of an $n\times n$ matrix A is the value of its determinant $\det(A)$. In particular, A is invertible if and only if no eigenvalue of A is zero. (Hint: Use Remainder Theorem.)
(b) (5 points) Let A be a self-adjoint $n\times n$ matrix and $g(x,y)=x^t Ay$ be the quadratic form associated to A. Prove that g is positive-definite if and only if A is positive-definite. (Hint: Use diagonalization.)
(c) (5 points) Let A and B be two positive-definite $n\times n$ matrices. Prove that $A+B$ is also positive-definite. In particular, $A+B$ is invertible.
(d) (5 points) Prove that the positive square root of any positive-definite $n\times n$ matrix C is unique. In other words, if A and B are two positive-definite $n\times n$ matrices such that $A^2=B^2=C$ then $A=B$.

國立中山大學101學年度碩士暨碩士專班招生考試試題

科目：機率論【應數系碩士班甲組】 題號：4054

答題時，每題須寫下題號與詳細步驟。 請依題號順序作答，不會作答題目請寫下題號並留空白。

1. Suppose the probability of picking a winning house in a race is 0.2, and X is the number of winning pick out of 10 races.
   (a) (5%) Find $P(X = 3)$.
   (b) (10%) Find the moment generating function of X.

2. Let $U, X_1, X_2, \ldots, X_n$ be independent uniform random variables on [0,1]. Denote the ordered random variables X_i by $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$.
   (a) (5%) Find $P(U \leq X_{(n)})$.
   (b) (10%) Find $P(X_{(1)} < U < X_{(n)})$.

3. Suppose that X and Y have the joint density function
$$f(x,y) = \frac{3}{2\pi}\sqrt{1-x^2-y^2}, \qquad x^2+y^2<1.$$
   (a) (10%) Find $P(X^2+Y^2 \leq 1/2)$.
   (b) (10%) Find the marginal density of X.
   (c) (5%) Find the conditional density of X given $Y = y$.

4. Let $X_1, X_2, \ldots$ be independent and identically beta distributed random variables with probability density function $f(x) = 12x(1-x)^2$, $0 < x < 1$. Let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.
   (a) (5%) Find $E\left[\frac{1}{X_1+1}\right]$.
   (b) (10%) Find the conditional expectation of Y_n given $X_1 = 3/5$.
   (c) (10%) Find the limiting distribution of Y_n.

5. (10%) Jolin and Jay alternate rolling a pair of fair dice, stopping either when Jolin rolls the sum 5 or when Jay rolls the sum 8. Assuming that Jolin rolls first, find the probability that the final roll is made by Jolin.

6. (10%) Suppose an ordinary deck of 52 cards is shuffled and the cards are then turned over one at a time until the first ace appears. Given that the first ace is the 22th card to appear, find the conditional probability that the card following it is the two of spades.

~ 全卷完 ~

國立中山大學 101 學年度碩士暨碩士專班招生考試試題

科目：數值分析【應數系碩士班乙組】

題號：4053
共 1 頁 第 1 頁

Please write down all the detail of your computation and answers.

1. (20%) Use (1) Lagrange formula, (2) Neville's method, and (3) Newton divided difference formula to compute the cubic polynomial $p(x)$ interpolating the following data

x	-1	0	1	2
y	4	3	2	7

2. (20%) (1) State the secant method, Newton method and the fixed point method to find a root of a given nonlinear equation. (2) State the advantage, disadvantage and the order of convergence of each method.

3. (20%) Derive the midpoint rule and composite midpoint rule for numerical integration with error formula.

4. (20%) Assume the Gaussian elimination on $n \times n$ matrix A needs no row exchange. Write a program to compute the LU (triangular) factorization of A using the least memory. How much storage is needed to run your program?

5. (20%) Let A be an $n \times n$ nonsingular matrix, $\mathbf{x}$ and $\hat{\mathbf{x}}$ be the exact and numerical solutions of linear system $A\mathbf{x} = \mathbf{b}$ respectively, and the residual $\mathbf{r} = \mathbf{b} - A\hat{\mathbf{x}}$. Show that

$$\frac{1}{\kappa(A)} \frac{||\mathbf{r}||}{||\mathbf{b}||} \leq \frac{||\hat{\mathbf{x}} - \mathbf{x}||}{||\mathbf{x}||} \leq \kappa(A) \frac{||\mathbf{r}||}{||\mathbf{b}||},$$

where $\kappa(A)$ is the condition number of A. Interpret the meaning of this error analysis and explain why it is important.

國立中山大學101學年度碩士暨碩士專班招生考試試題

科目：高等微積分【應數系碩士班丙組】

題號：4050
共 1 頁　第 1 頁

每題佔20%，總分100%。答題時，每題須寫下題號與詳細步驟。

1. Let $f: \mathbf{R}^2 \to \mathbf{R}$ be continuous.

   (a) (10%) Prove or disprove: f maps open sets in $\mathbf{R}^2$ to open sets in $\mathbf{R}$.

   (b) (10%) Prove or disprove: f is differentiable on $\mathbf{R}^2$.

2. (a) (6%) Find the Taylor polynomial $P_n(x)$ of the function $f(x) = \frac{1}{1+x}$ at the point $x_0 = 0$. Give the error term R_n also.

   (b) (6%) Show that for any $|x| < 1$, $\lim_{n\to\infty} R_n(x) = 0$.

   (c) (8%) Hence or otherwise, show that $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ when $|x| \leq 1$.

   (You may assume without proof that $\tan^{-1}(x) = \int_0^1 \frac{1}{1+x^2}\, dx$.)

3. Let $f: [a,b] \to \mathbf{R}$ be Lipschitz, that is, there is some $K > 0$ such that $|f(x) - f(y)| \leq K|x-y|$ for any x, y in $[a,b]$.

   (a) (6%) Show that f is continuous at any point $x_0 \in [a,b]$.

   (b) (8%) If $P = \{a = x_0, x_1 \ldots, x_n = b\}$ is a partition of $[a,b]$, show that

   $$0 \leq U(f,P) - L(f,P) \leq K(b-a)\|P\|,$$

   where $U(f,P)$ and $L(f,P)$ are the upper sum and lower sum of f with respect to P, and $\|P\| := \max_{1\leq k\leq n}(x_k - x_{k-1})$ is the norm of P.

   (c) (6%) Hence or otherwise, show that f is integrable on $[a,b]$.

4. Let $f: \mathbf{R}^n \to \mathbf{R}$ be differentiable at $\mathrm{a} \in \mathbf{R}^n$, with $f(\mathrm{a}) > 0$.

   (a) (10%) Show that there is some $\delta > 0$ such that $f(\mathrm{a}+\mathrm{h}) > 0$ for any $\|\mathrm{h}\| < \delta$.

   (b) (10%) Show from definition that the function $g(\mathrm{x}) = \frac{1}{f(\mathrm{x})}$ is differentiable at a with $\nabla g(\mathrm{a}) = \frac{-\nabla f(\mathrm{a})}{f(\mathrm{a})^2}$.

5. Let $f: \mathbf{R} \to \mathbf{R}$ be continuous such that $\lim_{x\to\infty} f(x) = L_+$, $\lim_{x\to-\infty} f(x) = L_-$, $(L_+, L_- \in \mathbf{R})$.

   (a) (10%) Prove or disprove: f achieves its maximum value.

   (b) (10%) Prove or disprove: If f is also differentiable on $\mathbf{R}$, then $\lim_{x\to\infty} f'(x) = 0$.

～全卷完～

國立中山大學101學年度碩士暨碩士專班招生考試試題

科目：微積分【應數系碩士班乙組】

題號：4051
共1頁第1頁

共8題。答題時，每題都必須寫下題號與詳細步驟。

[1]. (10%) Evaluate $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\tan x}\right)$.

[2]. (12%) Let $f(x)=2e^{g(x)}$ and $g(x)=\int_4^{x^2}\frac{2t}{1+t^4}dt$. Compute $f'(2)$.

[3]. Solve the following differential equations.

(a) (10%) $xy'=2y+x^3\sin x,\quad y(\pi)=0$.

(b) (10%) $\left(2x^2+y^2\right)dx+xydy=0,\quad y(1)=0$.

[4]. (12%) Evaluate the integral $\int_1^{\infty}(1-x)\,e^{-x}dx$.

[5]. (12%) Determine the interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{(x+3)^n}{n5^{n+1}}$.

[6]. (10%) Find the Maclaurin series for $f(x)=x^3\sin\left(\frac{x}{2}\right)$.

[7]. (12%) Evaluate the integral $\int_0^1\int_{4y^2}^{4}e^{\sqrt{x}}dxdy$.

[8]. (12%) Show that the Laplace equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$ in the coordinates (r,θ),

where $x=r^2\cos\theta$ and $y=r^2\sin\theta$ is $\frac{1}{r^4}\frac{\partial^2 f}{\partial\theta^2}+\frac{1}{4r^3}\frac{\partial f}{\partial r}+\frac{1}{4r^2}\frac{\partial^2 f}{\partial r^2}=0$.

======== 全卷完 ========

共十題，每題 10 分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. If X is an exponential random variable with mean $\frac{1}{\lambda}$, find that $E[X^k]$, $k = 1, 2, \ldots$

2. Find the probability density function of $Y = e^X$ when X is normally distributed with parameters μ and σ^2.

3. Let X_1, X_2, and X_3 be uncorrelated random variables, each with mean μ and variance σ^2. Find, in terms of μ and σ^2, $\mathrm{Cov}((X_1+X_2)(X_2+X_3))$ and $\mathrm{Cov}((X_1+X_2)(X_1-X_2))$

4. Let $X_1, \ldots, X_n$ be a random sample from a population with pdf

$$f_X(x) = \begin{cases} 1/\theta & 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent random variables.

5. Given that $N = n$, the conditional distribution of Y is χ^2_{2n}. The unconditional distribution of N is Poisson(θ). Calculate $E[Y]$ and $\mathrm{Var}(Y)$.

6. Let $X_1, \ldots, X_n$ be a random sample from the pdf

$$f(x|\mu,\sigma) = \frac{1}{\sigma}e^{-(x-\mu)/\sigma}, \quad \mu < x < \infty,\ 0 < \sigma < \infty.$$

Find a two-dimensional sufficient statistic for (μ, σ).

7. Let $X_1, \ldots, X_n$ be a random sample from a population with pdf

$$f(x|\theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \theta > 0.$$

(a) Is ΣX_i sufficient for θ?

(b) Find a complete sufficient statistic for θ.

8. Let $X_1, \ldots, X_n$ be a random sample from the pdf

$$f(x|\theta) = \theta x^{-2}, \quad 0 < \theta \le x < \infty.$$

(a) Find the MLE of θ.

(b) Find the method of moments estimator of θ.

9. Suppose that we have two independent random samples: $X_1, \ldots, X_n$ are exponential(θ), and $Y_1, \ldots, Y_m$ are exponential(μ). Find the likely ratio test (LRT) of $H_0\colon \theta = \mu$ versus $H_1\colon \theta \ne \mu$.

10. Derive a confidence interval for a binomial p by inverting the LRT of $H_0\colon p = p_0$ versus $H_1\colon p \ne p_0$.