國立中山大學96學年度碩士班招生考試試題

科目：工程數學【通訊所碩士班甲組】

1) (Totally, 15pts) Suppose that the random variable X has a Poisson distribution with parameter $\lambda > 0$
   (a) (5 pts) Find the moment generating function of X.
   (b) (5 pts) Show that $E(X) = \lambda$ and $Var(X) = \lambda$.
   (c) (5 pts) Show that
   $$P(X=0) \le P(X=1) \le \ldots \le P(X = \lfloor \lambda \rfloor)$$
   and
   $$P(X = \lfloor \lambda \rfloor) \ge P(X = \lfloor \lambda \rfloor + 1) \ge P(X = \lfloor \lambda \rfloor + 2) \ge \ldots,$$
   where $\lfloor y \rfloor$ is a floor function that returns the largest integer less than or equal to y.

2) (Totally, 20pts) Let $F_X(x)$ and $F_Y(y)$ be the distribution functions of X and Y, respectively, and let $F(x,y)$ be the joint distribution function of X and Y. Additionally, let $Z = \max(X,Y)$, $W = \min(X,Y)$.
   (a) (5 pts) Show that the distribution function of Z is $F(z,z)$ and the distribution function of W is $F_X(w) + F_Y(w) - F(w,w)$.
   (b) (5 pts) If $F(x,y)$ is continuous, find the densities of Z and W.
   (c) (10 pts) If X and Y are independent Gaussian random variables with $N(0,1)$, show that
   $$E(\max(X,Y)) = \frac{1}{\sqrt{\pi}}.$$

3) (Totally, 15pts) X is called a lognormal variable, if the $\log X = Y$ has a normal distribution $N(\mu, \sigma^2)$.
   (a) (5 pts) Find the density of X.
   (b) (5 pts) Find $E(X)$ and $Var(X)$.
   (c) (5 pts) Show that if the X_i are independent lognormal random variables, their product $X_1, X_2, \ldots, X_n$ is also lognormal.

4) (Totally, 10pts) Let $X_n = \sum_{i=1}^{n} a\cos\theta_i$ and $Y_n = \sum_{i=1}^{n} a\sin\theta_i$, where θ_i are independently uniform in $(0, 2\pi)$.
   (a) (5 pts) Show that the X_n and Y_n are uncorrelated but not independent.
   (b) (5 pts) Find the distribution of (X_n, Y_n) for large $n (n \to \infty)$.

5) (Totally, 20pts) Let V be a finite-dimensional inner product space, $T : V \to V$ be a projection, and $\|\cdot\|$ be the norm on V.
   (a) (10 pts) If T is an orthogonal projection, prove that $\|T(x)\| \le \|x\|$ for all $x \in V$. If equality holds, what can be concluded about T.
   (b) (10 pts) If T is also normal and V is complex, prove that T must be an orthogonal projection.

6) (Totally, 20pts) Let A be an $n \times n$ matrix with characteristic polynomial
   $$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$
   (a) (10 pts) Prove that A is invertible if and only if $a_0 \ne 0$.
   (b) (5 pts) Prove that if A is invertible, then
   $$A^{-1} = \frac{-1}{a_0}\left[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n\right].$$
   (c) (5 pts) Use part (b) to compute A^{-1} for
   $$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

1. (10%) If x(t) is a Gaussian pulse, show that the Fourier transform X(f) is also a Gaussian pulse.
2. (20%) x(t) is a random signal with autocorrelation function $R_{xx}(\tau)$ and power spectrum $S_{xx}(f)$. The random signal is applied to a filter with frequency response H(f) and impulse response h(t). (a) Derive the autocorrelation $R_{yy}(\tau)$ of the filter's output. (b) Derive the power spectrum $S_{yy}(f)$ of the filter's output.
3. (20%) Find the minimum required bandwidth and bit error rate for a noncoherently detected orthogonal binary FSK system with symbol duration T second.
4. (15%) Consider a carrier signal s at a frequency ω_0 and with an amplitude a, $s = a \cdot \exp(j\omega_0 t)$. The received signal s_r is the sum of n waves:

   $s_r = \sum_{i=1}^{n} a_i \exp[j(\omega_0 t + \theta_i)] \equiv r\exp[j(\omega_0 t + \theta)]$, where $r\exp(j\theta) = \sum_{i=1}^{n} a_i \exp(j\theta_i)$.

   Define: $r\exp(j\theta) = \sum_{i=1}^{n} a_i \cos\theta_i + j\sum_{i=1}^{n} a_i \sin\theta_i \equiv x + jy$

   We have: $x \equiv \sum_{i=1}^{n} a_i \cos\theta_i$ and $y \equiv \sum_{i=1}^{n} a_i \sin\theta_i$

   where: $r^2 = x^2 + y^2 \quad x = r\cos\theta \quad y = r\sin\theta$

   Assume (1) n is very large, (2) the individual amplitude a_i are random, and (3) the phase θ_i have a uniform distribution, it can be assumed that (from the central limit theorem) x and y are both Gaussian variables with means equal to zero and variance: $\sigma_x^2 = \sigma_y^2 \equiv \sigma^2$

   A. [10] Prove that the PDF of r is given by $p_R(r) = \begin{cases} \dfrac{r}{\sigma^2} e^{-r^2/2\sigma^2} & r \ge 0 \\ 0 & \text{otherwise} \end{cases}$.

   B. [5] Find the mean square value of r: $E\left[r^2\right]$.
5. (20%) Prove that if a signal $s(t)$ is corrupted by AWGN, the filter with an impulse response matched to $s(t)$ maximizes the output signal-to-noise ratio. The maximum SNR obtained with the matched filter is

   $$\text{SNR}_0 = \frac{2}{N_0}\int_0^T s^2(t)\,dt = \frac{2\varepsilon}{N_0}$$
6. (15%) The Hilbert transform is given by $\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$. Prove the following properties:

   A. [5] If $x(t) = x(-t)$, then $\hat{x}(t) = -\hat{x}(-t)$.

   B. [10] If $x(t) = \cos\omega_0 t$, then $\hat{x}(t) = \sin\omega_0 t$.

國立中山大學96學年度碩士班招生考試試題

科目：微分方程及向量分析【通訊所碩士班乙組】

1. A surface is given by

$$x^2+y^2+z^2=3.$$

(10%) (a) Find a unit vector perpendicular to the surface at the point (1,1,1).

(10%) (b) Derive the equation of the plane tangent to the surface at the point (1,1,1).

2. A vector is given by

$$\vec{F}=(x^2+y^2+z^2)^n(\vec{a}_x x+\vec{a}_y y+\vec{a}_z z).$$

(10%) (a) Find $\nabla\cdot\vec{F}$ and $\nabla\times\vec{F}$, respectively.

(10%) (b) Find a scalar potential $\varphi(x,y,z)$ such that $\vec{F}=-\nabla\varphi$ for the condition of $n=-1$ and $n\neq-1$, respectively.

3. Find a general solution for $X(x)$ in the following ordinary differential equations:

(10%) (a) $\dfrac{\mathrm{d}^2X(x)}{\mathrm{d}x^2}+X(x)=6\cos x+2$

(10%) (b) $x\dfrac{\mathrm{d}X(x)}{\mathrm{d}x}+X(x)=\dfrac{1}{x}$, for $x>0$

4. Solve the following initial-value problems using the Laplace transform.

(10%) (a) $\dfrac{\mathrm{d}f(t)}{\mathrm{d}t}-f(t)=H(t-1)$, where $f(0)=0$. Note that $H(t)$ is a unit step function.

(10%) (b) $\dfrac{\mathrm{d}^2f(t)}{\mathrm{d}t^2}+\dfrac{\mathrm{d}f(t)}{\mathrm{d}t}=1+\delta(t-2)$, where $f(0)=0$ and $\dfrac{\mathrm{d}f(t)}{\mathrm{d}t}=3$ at $t=0$. Note that $\delta(t)$ is a unit impulse function.

5. Two-dimensional Laplace's equation for scalar electric potential V in polar coordinates is given by

$$\nabla^2V(r,\phi)=0.$$

(10%) (a) Derive a general solution for $V(r,\phi)$ which has ϕ-dependence and vanishes as $r\to\infty$.

(10%) (b) Determine $V(r,\phi)$ by assuming no ϕ-dependence and specifying the boundary conditions, $V=V_0$ at $r=a$ and $V=0$ at $r=b$, for $a\le r\le b$.

1. A surface charge is distributed uniformly with density ρ_{s0} on a rectangular surface of sides a and b. Find the electric potential at the center of the rectangular surface. (15%)

2. A large conducting plate of thickness d is located at $-d/2 \le y \le d/2$, as shown in Fig. P2. A uniform current density $\mathbf{J} = \mathbf{a}_z J_0$ is flowing in the z-direction. Find the magnetic field in all regions. (15%)

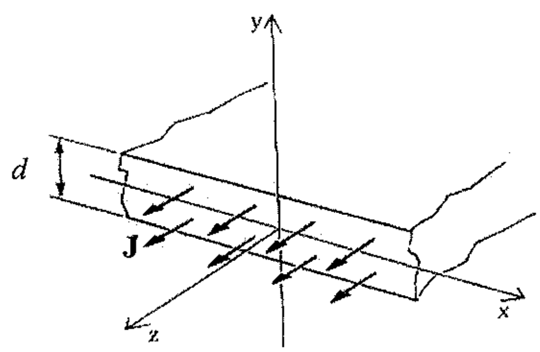

Fig. P2

3. The electric field component of an electromagnetic wave in free space is given by

$$\mathbf{E}(y,z,t) = \hat{\mathbf{a}}_x E_0 \cos(ay)\cos(\omega t - bz).$$

(a) Find the corresponding magnetic field $\mathbf{H}(y, z, t)$. (5%)
(b) Find the relationship between a, b and ω such that all of Maxwell's equations are satisfied. (5%)
(c) This wave can be regarded as the sum of two uniform plane waves. Determine the direction of propagation of these two component waves. (5%)

4. A 500 MHz uniform plane wave is normally incident on a freshwater (ε_r = 88) lake covered with a layer of ice (ε_r = 3.2), as shown in Fig. P.4.
(a) Find the minimum thickness of the ice such that the reflected wave has a maximum strength. Assume the lake water to be very deep. (5%)
(b) What is the ratio of the amplitude of the reflected and incident electric fields? (5%)

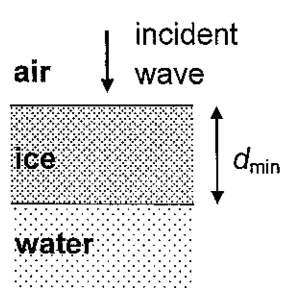

Fig. P4.

5. A 150-Ω transmission line is connected to two loads as shown in Fig. P.5. Find l_1 , l_2 , R_1 and R_2 such that the loads receive equal powers with in-phase voltages. (20%)

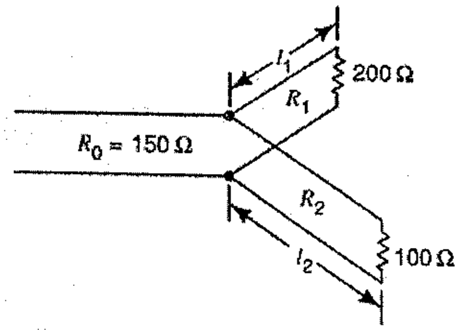

Fig. P5.

6. This problem describes the procedure to find the TM modes inside a metallic rectangular waveguide of dimensions $a \times b$, with $a > b$.
   (a) First, write down the time-harmonic Maxwell's equations. (5%)
   (b) Derive the equations expressing the transverse components of electric and magnetic fields in terms of the longitudinal components. (10%)
   (c) Solve the longitudinal component from the Helmholtz equation and related boundary conditions, and then the rest of the field components from the longitudinal component. (10%)