國立中山大學104學年度碩士暨碩士專班招生考試試題

科目名稱：線性代數【通訊所碩士班甲組】 題號：437002

※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘）（選擇題） 共4頁第1頁

For each of the following questions, please select <u>the best answer</u> from the choices provided. (單選) You do NOT need to provide any justification.

1. (5%) Suppose that y is the corresponding output of a system with x_1 and x_2 being the inputs. Let α_i for $i = 1, 2, \cdots$ be the coefficients of the system. Which of the following system is the linear system?
   (A) $y = \alpha_1 x_1 + \alpha_2 x_2^2 + \alpha_3 x_1^2$
   (B) $y = \alpha_1^2 x_1 + \alpha_2^2 x_2 + \alpha_3^3 x_1$
   (C) $y = \log(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)$
   (D) $y = \alpha_1 \sin(x_1) + \alpha_2 \cos(x_2) + \alpha_3 \tan(x_1)$
   (E) none of the above hold.

2. (5%) Suppose a 5 by 4 matrix $\mathbf{A}$ has rank 4. The equation $\mathbf{b} = \mathbf{A}\mathbf{x}$ for any $\mathbf{b} \in \mathbb{R}^5$
   (A) always has a unique solution.
   (B) always has no solution.
   (C) always has many solutions.
   (D) sometimes but not always has a unique solution.
   (E) sometimes but not always has many solutions.

3. (5%) Let $\mathbf{v}_1 = [1\ 2\ 3]^T, \mathbf{v}_2 = [4\ 5\ 6]^T, \mathbf{v}_3 = [2\ 1\ 0]^T$. Then
   (A) the set $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ is linearly independent.
   (B) the set $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ is linearly dependent.
   (C) the set $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ can span $\mathbb{R}^3$.
   (D) we cannot find $\alpha_1, \alpha_2, \alpha_3 \neq 0$ so that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$.
   (E) none of the above hold.

4. (5%) Let T be the linear transformation whose standard matrix is
   $$\mathbf{A} = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$
   Then
   (A) T maps $\mathbb{R}^4$ onto $\mathbb{R}^4$.
   (B) T maps $\mathbb{R}^4$ onto $\mathbb{R}^3$.
   (C) T is a one-to-one mapping.
   (D) we sometimes cannot find a solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ for $\mathbf{b} \in \mathbb{R}^3$.
   (E) none of the above hold.

5. (5%) The eigenvalues of
   $$\mathbf{A} = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
   are
   (A) $5, -2, 6, -1$.
   (B) $0, 3, -8, 0$.
   (C) $-2, -8, 4, 1$.
   (D) $5, 3, 5, 1$.
   (E) none of the above.

6. (5%) Suppose

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

Then

(A) $\mathbf{Ax} = \mathbf{b}$ has many solutions.
(B) $\mathbf{A}$ is invertible.
(C) the determinant of $\mathbf{A}$ is zero, i.e., $\det(\mathbf{A}) = 0$.
(D) 0 is one of the eigenvalue of $\mathbf{A}$.
(E) none of the above hold.

7. (5%) Let $\mathbf{A}$ be an invertible $n \times n$ matrix. Which of the following statement is false:
(A) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
(B) $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.
(C) There is an $n \times n$ matrix $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}$.
(D) $\mathbf{A}^T$ is an invertible matrix.
(E) The columns of $\mathbf{A}$ form a linearly dependent set.

8. (5%) The rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

is
(A) 0.
(B) 1.
(C) 2.
(D) 3.
(E) 4.

9. (5%) Let $\mathbf{A}$ be a square $n \times n$ matrix. Which of the following statement is not equivalent to others:
(A) The columns of $\mathbf{A}$ form a basis of $\mathbb{R}^n$.
(B) The rank of $\mathbf{A}$ is n, i.e., $\mathrm{rank}(\mathbf{A}) = n$.
(C) $\mathbf{A}$ is an invertible matrix.
(D) The dimension of null space of $\mathbf{A}$ is n, i.e., $\dim \mathrm{Nul}\mathbf{A} = n$.
(E) There is an $n \times n$ matrix $\mathbf{D}$ such that $\mathbf{AD} = \mathbf{I}$.

10. (5%) Suppose H is a subspace of $\mathbb{R}^n$. Which of the following property is not necessarily true:
(A) The zero vector is in H.
(B) For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
(C) For each $\mathbf{u}$ and $\mathbf{v}$ in H, the product $\mathbf{u} \odot \mathbf{v} = [u_1v_1 \; u_2v_2 \; \cdots \; u_nv_n]^T$ is in H.
(D) For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.
(E) A basis of H is a linearly independent set.

11. (5%) The determinant of

$$\mathbf{A} = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

is

(A) 0. (B) 24. (C) -6. (D) -12. (E) -24.

12. (5%) Let $\mathbf{A}$ be a square matrix. Which of the following statement is false:

(A) If a multiple of one row of $\mathbf{A}$ is added to another row to produce a matrix $\mathbf{B}$, then $\det(\mathbf{B}) = \det(\mathbf{A})$.

(B) If two rows of $\mathbf{A}$ are interchanged to produce $\mathbf{B}$, then $\det(\mathbf{B}) = \det(\mathbf{A})$.

(C) If one row of $\mathbf{A}$ is multiplied by c to produce $\mathbf{B}$, then $\det(\mathbf{B}) = c \cdot \det(\mathbf{A})$.

(D) $\det(\mathbf{A}^T) = \det(\mathbf{A})$.

(E) Let $\mathbf{B}$ be a square matrix. $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

13. (5%) Let

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 & -1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}.$$

If the column space of $\mathbf{A}$ is a subspace of $\mathbb{R}^k$, what is k? If the null space of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$, what is n?

(A) $(k,n) = (1,2)$.

(B) $(k,n) = (3,3)$.

(C) $(k,n) = (3,4)$.

(D) $(k,n) = (4,3)$.

(E) $(k,n) = (2,2)$.

14. (5%) Let H be a subspace of a finite-dimensional vector space V. Which of the following statement is false:

(A) If a vector space V has basis $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

(B) Any linearly independent set in H can be expanded to a basis for H.

(C) $\dim H \leq \dim V$.

(D) Let V be a n-dimensional vector space, $n \geq 1$. Any linearly independent set of exactly n elements in V spans V.

(E) If $\dim V = n$ and if H is a linearly dependent subset of V, then H contains more than n vectors.

15. (5%) Let $\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Which of the following vector is the eigenvector of $\mathbf{A}$

(A) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (B) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. (C) $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$. (D) $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(E) none of the above.

16. (5%) Let u, v, and w be the vector in $\mathbb{R}^n$, and let c be a scalar. The inner product of u and v is written as $\mathrm{u}\cdot\mathrm{v}$. The length of u is defined by $\|\mathrm{u}\| = \sqrt{\mathrm{u}\cdot\mathrm{u}}$. Which of the following statement is false:
(A) $\mathrm{u}\cdot\mathrm{v} = \mathrm{v}\cdot\mathrm{u}$.
(B) $(\mathrm{u}+\mathrm{v})\cdot\mathrm{w} = \mathrm{u}\cdot\mathrm{w} + \mathrm{v}\cdot\mathrm{w}$.
(C) $(c\mathrm{u})\cdot\mathrm{v} = \mathrm{u}\cdot(c\mathrm{v})$.
(D) $\|\mathrm{u}\|^2 = 0$ if and only if $\mathrm{u} = 0$.
(E) $\|\mathrm{u}+\mathrm{v}\|^2 = \|\mathrm{u}\|^2 + \|\mathrm{v}\|^2$.

17. (5%) An orthogonal matrix is a square invertible matrix U such that $\mathrm{U}^{-1} = \mathrm{U}^T$. Which of the following statement is false:
(A) Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.
(B) A matrix with orthonormal columns is an orthogonal matrix.
(C) If the column of an $m \times n$ matrix A are orthonormal, then the linear mapping $\mathrm{x} \mapsto \mathrm{Ax}$ preserves length.
(D) An orthogonal matrix is invertible.
(E) The orthogonal projection of y onto v is the same as the orthogonal projection of y onto $c\mathrm{v}$ whenever $c \neq 0$.

18. (5%) Which of the following statement is false:
(A) If z is orthogonal to u_1 and to u_2 and if $W = \mathrm{Span}\{\mathrm{u}_1\mathrm{u}_2\}$, then z must be in $W^\perp$.
(B) For each y and each subspace W, the vector $\mathrm{y} - \mathrm{proj}_W\mathrm{y}$ is orthogonal to W, where $\mathrm{proj}_W\mathrm{y}$ denotes the projections of y onto W.
(C) The orthogonal projection $\hat{\mathrm{y}}$ of y onto a subspace W can sometimes depend on the orthogonal basis for W used to compute $\hat{\mathrm{y}}$.
(D) If y is in a subspace W, then the orthogonal projection of y onto W is y itself.
(E) If the columns of an $n \times p$ matrix U are orthonormal, then $\mathrm{UU}^T\mathrm{y}$ is the orthogonal projection of y onto the column space of U.

19. (5%) Which of the following statement is false:
(A) If $\mathrm{Ax} = \lambda\mathrm{x}$ for some vector x, then λ is an eigenvalue of A.
(B) If $\mathrm{Ax} = \lambda\mathrm{x}$ for some vector $\mathrm{x} \neq 0$, then x is an eigenvector of A.
(C) A matrix A is not invertible if and only if 0 is an eigenvalue of A.
(D) A number c is an eigenvalue of A if and only if the equation $(\mathrm{A} - c\mathrm{I})\mathrm{x} = 0$ has a nontrivial solution.
(E) Finding an eigenvector of A may be difficult, but checking whether a given vector is an eigenvector is easy.

20. (5%) Let matrix A and B be $n \times n$ matrices. Which of the following statement is true:
(A) The determinant of A is the product of the diagonal entries in A.
(B) The trace of A is the sum of the diagonal entries in A.
(C) If $\lambda + 5$ is a factor of the characteristic polynomial of A, then 5 is an eigenvalue of A.
(D) An elementary row operation on A does not change the determinant.
(E) A row replacement operation on A does not change the eigenvalues.