國立中山大學 102 學年度碩士暨碩士專班招生考試試題

科目名稱：線性代數【通訊所碩士班甲組】 題號：437002
※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘） 共 2 頁第 1 頁

1. (20 %) For each of the following statements, mark "**O**" if the statement is TRUE and "**X**" if the statement is FALSE. You do NOT need to provide any justification.
   (a) (   ). The fixed points **u** of a transformation $T(\mathbf{u}) = \mathbf{Bu}$ are eigenvectors with its eigenvalue being one.
   (b) (   ). The set of all vectors of the form $(3a+b+1, 2a, b)$ is a subspace of $\mathbb{R}^3$.
   (c) (   ). The dimension of $\text{Span}\{e^t, e^{3t}, 2e^t + 3e^{3t}, e^t - 2e^{3t}\}$ is 4.
   (d) (   ). The eigenvalues of a square matrix must be distinct.
   (e) (   ). Let two vectors be $\mathbf{u} = \begin{bmatrix} i \\ 6 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2i \\ 0 \\ 1 \end{bmatrix}$. Then, $\mathbf{u} \perp \mathbf{v}$.

2. (10%) Let **A** and **B** be 3x3 matrices with det(**A**) = 5, det(**B**) = 10 and det(**A** + **B**) = 60. Decide the following values.
   (a) $\det(\mathbf{A} + \mathbf{A})$.
   (b) $\det(\mathbf{A}^2\mathbf{B} + \mathbf{AB}^2)$.

3. (20%) Define $T: P_2 \to \mathbb{R}^3$ by $T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$ where $p(t)$ in P_2 can be expressed as $p(t) = at^2 + bt + c$.
   (a) Find the image under T when $p(t) = 6 + 2t$.
   (b) Show that T is a linear transformation.
   (c) Find the kernel of T.
   (d) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for P_2. (This means that the matrix will act on the coordinates of p).

4. (10%) A square matrix is called upper triangular if all the entries below the main diagonal are zero. The product of upper triangular matrices is
   (a) lower triangular matrix,
   (b) upper triangular matrix,
   (c) diagonal matrix.

   Please select the best answer and you do NOT need to provide any justification.

5. (10%) Which of the following is not a linear equation of (x_1, x_2, x_3)?
   (a) $x_1 + 4x_2 + 1 = x_3$
   (b) $x_1 = 1$
   (c) $x_1 + 4x_2 - \sqrt{2}x_3 = \sqrt{4}$
   (d) $x_1 + 4x_1x_2 - \sqrt{2}x_3 = \sqrt{4}$

   Please select the best answer and you do NOT need to provide any justification.

6. (10%) If $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, then the eigenvalues of $\mathbf{A}$ is

(a) $(1, 2, -1)$).

(b) $(1, 1/2, -1)$.

(c) $(1, -2, -1)$.

(d) $(1, -1/2, -1)$.

Please select the best answer and you do NOT need to provide any justification.

7. (10%) If $\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ -2 & -4 & -9 \end{bmatrix}$ and $\det(\mathrm{A}) = 0$, then $\mathrm{rank}(\mathrm{A})$ is

(a) 0.

(b) 1.

(c) 2.

(d) 3.

Please select the best answer and you do NOT need to provide any justification.

8. (10%) Let $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4$ be four different vectors in $\mathbb{R}^3$. Then

(a) they must be linearly independent.

(b) they must be linearly dependent.

(c) they must be either linearly independent or linearly dependent.

(d) none of the above hold.

Please select the best answer and you do NOT need to provide any justification.

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科目名稱：機率【通訊所碩士班甲組】　　　　　　　　　　　　　　　　題號：437004

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1. (Totally, 10 pts) The conditional probability density of X provided that the continuous event V has values between y and $y+dy$ is given by

$$p_{X|Y}(x\,|\,y)=\frac{2xy+a}{a^2(y+1)} \text{ for } 0\le x\le a,$$

Where a is a constant and the probability density of Y is given by

$$p_Y(y)=\frac{2(y+1)}{b^2+2b} \text{ for } 0\le y\le b.$$

Please find the conditional probability density function $p_{Y|X}(y\,|\,x)$, i.e., the conditional probability density function of Y provided that the continuous event U has values between x and $x+dx$.

2. (Totally, 10 pts) Let Z_1 and Z_2 be independent and have exponential distribution with density $\lambda e^{-\lambda z}$ for $z\ge 0$. Define $X=Z_2$ and $Y=Z_1+Z_1Z_2$. Please find $E[E[Y|X]]$.

3. (Totally, 15 pts) Markov Inequality is expressed as follows. Let Y be a non-negative random variable with finite expectation $E[Y]=\eta$, then, for any $\alpha>0$,

$$P\{Y>\alpha\}\le\frac{\eta}{\alpha}.$$

(a) (5 pts) Please prove Chernoff Bound using Markov Inequality. Note that Chernoff Bound is given by, for a random variable X,

$$P\{X>\alpha\}\le\frac{E[e^{sX}]}{e^{s\alpha}}, \text{ for } s>0$$

(b) (10 pts) The characteristic function of a Gaussian random variable X distributed as $N(\mu,\sigma^2)$ is given by

$$\Phi_X(\omega)=e^{j\mu\omega-\frac{\sigma^2\omega^2}{2}}.$$

Please find the Chernoff Bound for the above Gaussian random variable.

4. (Totally, 15 pts) Let X and Y be independent random variables each Poisson distributed with parameter λ.

(a) (5 pts) Find the probability mass function of $X+Y$.

(b) (5 pts) Find the distribution function of $\min(X,Y)$.

(c) (5 pts) Find the conditional probability $P(Y=y|X+Y=z)$ for $y=0,1\ldots,z$.

5. (Totally, 15 pts) Consider a random variable X with the following PDF

$$p(x)=\tfrac{3}{2}x^2, \quad \text{for } -1\le x\le 1$$

(a) (5pts) Plot the cumulative distribution function (CDF) of X

(b) (5pts) Find the expectation $E[X]$

(c) (5pts) Find the variance of X

6. (Totally, 25 pts) Let X and Y be two random variables with the following joint PMF:

$$P(X,Y) = \begin{cases} \frac{3}{10}, & X=1, Y=2 \\ \frac{1}{10}, & X=1, Y=4 \\ \frac{1}{5}, & X=2, Y=2 \\ \frac{2}{5}, & X=2, Y=4 \end{cases}$$

(a) (5pts) Find the conditional probability $P(X|Y=2)$.
(b) (5pts) Find the conditional expectation $E[X|Y=4]$
(c) (5pts) Are X and Y independent of each other? Please prove your answer.
(d) (10pts) Find the correlation coefficient between X and Y.

7. (Totally, 10 pts) Let U be a random variable uniformly distributed between 0 and 1. Answer the following questions.
(a) (5pts) For any strictly increasing function $f : \mathbb{R} \to [0,1]$, find the CDF of $X = f^{-1}(U)$.
(b) (5 pts) Given any random variable X with PDF $p(x)$, show that X can be generated by $X = f^{-1}(U)$, where

$$f(x) = \int_{-\infty}^{x} p(v)\ dv.$$

國立中山大學 102 學年度碩士暨碩士專班招生考試試題

科目名稱：通訊理論【通訊所碩士班甲組】　　　　　　　　　　　　　題號：437005
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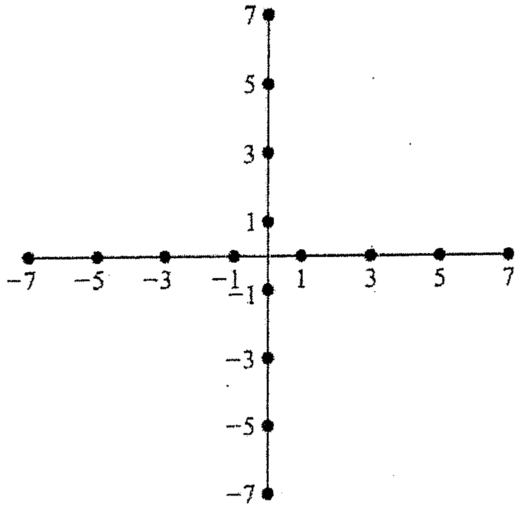
1. [20] Signal Constellation and Decision Boundaries

For the QAM signal constellation shown in the following figure, determine the optimum decision boundaries for the detector, assuming that the signal to noise ratio (SNR) is sufficiently high so that errors only occur between adjacent points.



2. [20] Random Process

Please prove that if the input to a stable linear time-invariant filter is a wide-sense stationary random process, then the output of the filter is also wide-sense stationary.

3. [20] Characteristic Function and Gaussian Random Variable

The characteristic function of a random variable X is defined as the statistical average: $E(e^{jvX}) \equiv \psi(jvX) = \int_{-\infty}^{\infty} e^{jvx} p(x)dx$.

A. [10] Find the characteristic function of the Gaussian random variable.

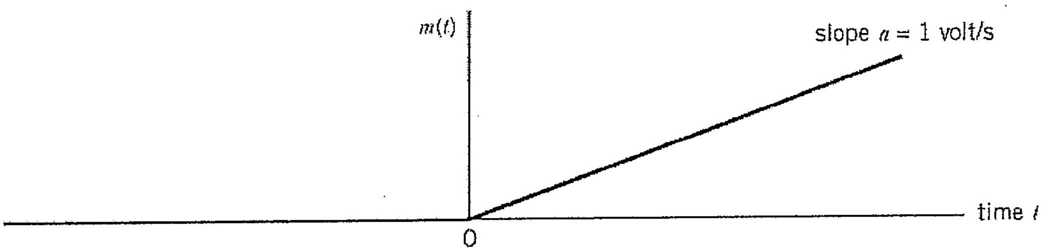
B. [10] Show that the sum Y of N independent and identically distributed (i.i.d.) Gaussian random variables, X_i, $i = 1, 2, \ldots, N$, is a Gaussian random variable.

4. [20] Phase/Frequency Modulation

For the message signal $m(t)$ shown in the following figure:

A. [10] Please draw a diagram to illustrate the phase modulated wave.

B. [10] Please draw a diagram to illustrate the frequency modulated wave.



國立中山大學 102 學年度碩士暨碩士專班招生考試試題

科目名稱：通訊理論【通訊所碩士班甲組】 題號：437005

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5. [20] Fourier Transform:

A. [10] The Fourier transform of a decaying exponential pulse is given by: $\exp(-at)u(t) \rightleftharpoons \frac{1}{a+j2\pi f}$, $a > 0$, where $u(t)=\begin{cases}1, & t>0\\ \frac{1}{2}, & t=0\\ 0, & t<0\end{cases}$. Please show that the Fourier transform of a double exponential pulse is given by: $\exp(-a|t|) \rightleftharpoons \frac{2a}{a^2+(2\pi f)^2}$, $a>0$.

B. [5] Please show that the Fourier transform of a signum function is given by $\operatorname{sgn}(t) \rightleftharpoons \frac{1}{j\pi f}$, where $\operatorname{sgn}(t)=\begin{cases}+1, & t>0\\ 0, & t=0\\ -1, & t<0\end{cases}$.

C. [5] Please find the Fourier transform of a unit step function $u(t)$.

Property	Mathematical Description
Linearity	$ag_1(t)+bg_2(t) \rightleftharpoons aG_1(f)+bG_2(f)$, where a and b are constants.
Time scaling	$g(at) \rightleftharpoons \frac{1}{\lvert a\rvert}G\left(\frac{f}{a}\right)$, where a is a constant.
Duality	If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$.
Time shifting	$g(t-t_0) \rightleftharpoons G(f)\exp(-j2\pi ft_0)$.
Frequency shifting	$\exp(j2\pi f_c t)g(t) \rightleftharpoons G(f-f_c)$.
Area under $g(t)$	$\int_{-\infty}^{\infty} g(t)\,dt = G(0)$.
Differentiation in the time domain	$\frac{d}{dt}g(t) \rightleftharpoons j2\pi fG(f)$.
Integration in the time domain	$\int_{-\infty}^{t} g(\tau)\,d\tau \rightleftharpoons \frac{1}{j2\pi f}G(f)+\frac{G(0)}{2}\delta(f)$.
Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$.
Multiplication in the time domain	$g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G(f-\lambda)\,d\lambda$.
Convolution in the time domain	$\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)\,d\tau \rightleftharpoons G_1(f)G_2(f)$.
Rayleigh's energy theorem	$\int_{-\infty}^{\infty} \lvert g(t)\rvert^2 dt = \int_{-\infty}^{\infty} \lvert G(f)\rvert^2 df$.