國立中山大學 99 學年度博士班招生考試試題

科目：機率論【應數系博士班甲組】

Do all problems in detail. 20 points for each problem.

1. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables with $P(X_1 = 1) = P(X_1 = -1) = 1/2$. Let $S_n = n^{-1/2}\sum_{i=1}^{n} X_i$. Find $\lim_{n\to\infty} P(-1 < S_n < 1)$.

2. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables such that for $n \geq 1$, $P(X_n = 1) = p_n = 1 - P(X_n = 0)$, $0 \leq p_n \leq 1$, and let $Y_n = \sum_{i=1}^{n}(X_i - p_i)/n$. Show that Y_n converges in probability to 0.

3. For any random variable X, a real number c is called a median of X if $P(X < c) \leq \frac{1}{2} \leq P(X \leq c)$. Show that if $P(|X| > 1) < 1/2$, then any median of X must lie in the interval $[-1, 1]$.

4. Let X and Y be independent and identically distributed random variables. If X is exponentially distributed with parameter 1, show that $X/(X+Y)$ and $X+Y$ are independent.

5. Let X be a non-negative random variable. Show that

$$\sqrt{1+(E[X])^2} \leq E\sqrt{1+X^2} \leq 1 + EX.$$

~ End of Paper ~

每題 20 分，共 5 題，請詳列計算和推導過程書寫於題目下方空白處。

問題	1: 20 分	2: 20 分	3: 20 分	4: 20 分	5: 20 分	總分: 100 分
得分						

1. Let A, B, and C be independent random variables, uniformly distributed on $(0, 1)$. What is the probability that $Ax^2 + Bx + C$ has real roots?

2. Let X_1, X_2, ... be iid and $X_{(n)} = \max_{1 \le i \le n} X_i$.

   (a) If X_i is beta$(1, \beta)$, find a value of ν so that $n^{\nu}(1 - X_{(n)})$ converges in distribution.

   (b) If X_i is exponential(1), find a sequence a_n so that $X_{(n)} - a_n$ converges in distribution.

3. Let $(X_1, \ldots, X_n)$ be a sample of Bernoulli random variables with $P(X_i = 1) = p \in (0, 1)$. Find the UMVUE of p^m, where m is a positive integer and $m \leq n$.

4. Let $(X_1, \ldots, X_n)$ be a random sample from the following discrete distribution:

$$P(X_1 = 1) = \frac{2(1-\theta)}{2-\theta}, \quad P(X_1 = 2) = \frac{\theta}{2-\theta},$$

where $\theta \in (0,1)$ is unknown. Obtain a moment estimator of θ and its asymptotic distribution.

5. Suppose that $X_1, \ldots, X_n$ are iid with a beta$(\mu, 1)$ pdf and $Y_1, \ldots, Y_m$ are iid with a beta$(\theta, 1)$ pdf. Also assume that the Xs are independent of the Ys.

   (a) Find an LRT of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$.

   (b) Show that the test in part (a) can be based on the statistic

$$T = \frac{\sum \log X_i}{\sum \log X_i + \sum \log Y_i}.$$

~全卷完~

國立中山大學 99 學年度博士班招生考試試題

科目：分析【應數系博士班丙組】

Department of Applied Mathematics at National Sun Yat-sen University
PhD Entrance Exam Analysis Question paper (May 2010)

(Note: This question paper is composed of five (5) questions, each of which carries 20 points, with a total of 100 points. Attempt all of them.)

Question One (20 points)
Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers.

(1.1) If the following condition is satisfied:

$$a_{m+n} \leq a_m + a_n \quad m, n \geq 0,$$

prove that

$$\lim_{n\to\infty} \frac{a_n}{n} \quad \text{exists.}$$

(1.2) Suppose that the following condition is satisfied:

$$a_{n+1} \leq a_n + \delta_n, \quad n \geq 0,$$

where $\{\delta_n\}$ is a sequence of nonnegative real numbers such that $\sum_{n=0}^{\infty} \delta_n < \infty$. Prove that $\lim_{n\to\infty} a_n$ exists.

Question Two (20 points)
Let

$$f_n(x) = \frac{nx - 1}{(x \log n + 1)(1 + nx^2 \log n)}.$$

Show that

(2.1) $\lim_{n\to\infty} f_n(x) = 0, \;\; 0 < x \leq 1.$

(2.2) $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \frac{1}{2}.$

Question Three (20 points)
Suppose $f \in L^1(-\infty, +\infty)$ and let

$$F_n(x) = \frac{n}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)dt}{1 + n^2(t-x)^2}.$$

Show that $\lim_{n\to\infty} F_n(x) = f(x)$ if f is continuous at x.

Question Four (20 points)
If f is integrable over a finite closed interval $[a, b+\delta]$ (where $\delta > 0$), then

$$\lim_{h\to 0^+} \int_a^b |f(x+h) - f(x)|dx = 0.$$

Question Five (20 points)
Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let f be an integrable function on Ω and let $\{f_n\}$ be a sequence of integrable functions on Ω. Prove that $f_n \to f$ in L^1, i.e.,

$$\lim_{n\to\infty} \int_\Omega |f_n(\omega) - f(\omega)|dP(\omega) = 0$$

if and only if

$$\lim_{n\to\infty} \int_A f_n(\omega)dP(\omega) = \int_A f(\omega)dP(\omega) \quad \text{holds uniformly over } A \in \mathcal{F}.$$

-End-