(1) Show that for any $r > 1$, $E[|\mathbf{X}|^r] < \infty$ if and only if

$$\sum_{n=1}^{\infty} n^{r-1}\mathbf{Pr}(|\mathbf{X}| \geq n) < \infty. \qquad (15\%)$$

(2) Suppose that $\mathbf{X}_n \to \mathbf{X}$ and $\mathbf{Y}_n \to \mathbf{Y}$ both in probability. Prove or disprove: $\mathbf{X}_n + \mathbf{Y}_n \to \mathbf{X} + \mathbf{Y}$ in probability. (10%)

(3) Let $\mathbf{X}$ be a random variable such that $M_{\mathbf{X}}(t) = E[e^{t\mathbf{X}}]$ is finite for all t. Show that

$$\mathbf{Pr}(\mathbf{X} \geq a) \leq e^{-ta}M_{\mathbf{X}}(t), \quad t > 0. \qquad (10\%)$$

(4) Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are random variables. If $\mathbf{X}^2$ and $\mathbf{Y}^2$ are independent. Are $\mathbf{X}$ and $\mathbf{Y}$ necessarily independent? (15%)

(5) Suppose that $\{\mathbf{X}_n\}$ converges in distribution to $\mathbf{X}$. For each $p \geq 1$, prove or disprove: $E[|\mathbf{X}_n|^p] \to E[|\mathbf{X}|^p]$. (15%)

(6) Let $\mathbf{X}_n$, $n = 1, 2, \ldots$, be i.i.d. random variables with $\mathbf{Pr}(X_1 = n) = \mathbf{Pr}(X_1 = -n) = c(n^2 \log n)^{-1}$, $n = 3, 4, \ldots$, where $c = [2\sum_{n=3}^{\infty}(n^2 \log n)^{-1}]^{-1}$. Let $S_n = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n$. Does $S_n/n \to 0$ in probability? Does $S_n/n \to 0$ almost surely? (20%)

(7) Let $1 \leq p < \infty$, $\mathbf{X}_n \in L^p$, and $\mathbf{X}_n \to \mathbf{X}$ in probability. Show that if $\{|\mathbf{X}_n|^p\}$ is uniformly integrable then $\mathbf{X}_n \to \mathbf{X}$ in L^p. (15%)

國立中山大學97學年度博士班招生考試試題

科目：**數理統計【應數系甲組】**

共五題，每題20分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. Let X and Y be independent $N(0,1)$ random variables, and define a new random variable Z by

$$Z = \begin{cases} X & \text{if } XY > 0, \\ -X & \text{if } XY < 0. \end{cases}$$

   (a) Show that Z has a normal distribution.

   (b) Show that the joint distribution of Z and Y is not bivariate normal.

2. Let $X_1, \ldots, X_n$ be a random sample from the pdf

$$f(x|\mu,\sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, \quad \mu < x < \infty, \ 0 < \sigma < \infty.$$

   Find a two-dimensional sufficient statistic for (μ, σ).

3. Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x|\theta) = \frac{1}{\theta}, \quad 0 \le x \le \theta, \ \theta > 0.$$

   Estimate θ using both the method of moments and maximum likelihood. Calculate the means and variances of the two estimators. Which one should be preferred and why?

4. Suppose that we have two independent random samples: $X_1, \ldots, X_n$ are exponential(θ), and $Y_1, \ldots, Y_m$ are exponential(μ).

   (a) Find the likelihood ratio test (LRT) of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$.

   (b) Show that the test in part (a) can be based on the statistic

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_i}.$$

   (c) Find the distribution of T when H_0 is true.

5. (a) Find the $1-\alpha$ confidence set for a that is obtained by inverting the LRT of $H_0 : a = a_0$ versus $H_1 : a \neq a_0$ based on a sample $X_1, \ldots, X_n$ from a $N(\theta, a\theta)$ family, where θ is unknown.

   (b) A similar question can be asked about the related family, the $N(\theta, a\theta^2)$ family, If $X_1, \ldots, X_n$ are iid $N(\theta, a\theta^2)$, where θ is unknown, find the $1-\alpha$ confidence set based on inverting the LRT of $H_0 : a = a_0$ versus $H_1 : a \neq a_0$.

～全卷完～

科目：數值分析【應數系乙組】

Entrance Exam for the Ph.D Program of Scientific Computing

Six questions with the marks indicated.

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1. (10) For numerical methods, take one kind of numerical methods,

(a) Give the definitions of convergence and stability, (b) What are the differences between convergence and stability.

2. (10) Prove the Schwarz inequality:

$$|\sum_{i=1}^{n} x_i y_i| = \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}.$$

Note that you can not use the formula: $(x, y) = \|x\|\,\|y\|\cos(x, y)$ to prove the Schwarz inequality.

3. (15) To seek a root of $x = F(x)$, choose the substitution iteration: $x_{k+1} = F(x_k)$, $k = 0, 1, ...$ Suppose that $|F'(x)| < 1$. (a) To prove the solution existence. (b) To derive the error bounds.

4. (15) Consider the initial value problem of ordinary differential equations (ODEs),

$$y' = f(x, y), \ x \geq 0; \ y(0) = y_0.$$

Give the midpoint scheme, derive the local errors and provide stability analysis.

5. (15) Choose the central and the trapezoidal rules to evaluate the integral $\int_a^b f(x)dx$. Suppose that $f''(x)$ exists and $f''(x) \geq 0$ (or $f''(x) \geq 0$). Prove that the numerical values by the central and the trapezoidal rules are just the upper and lower bounds.

6. (35) Solve the linear algebraic equations $Ax = b$, where the matrix $A \in R^{n\times n}$ and vectors $x \in R^n$ and $b \in R^n$. For the perturbation equations $A(x + \Delta x) = b + \Delta b$, where $\Delta b \in R^n$ and $\Delta x \in R^n$.

(a) To prove the bound,

$$\frac{\|\Delta x\|}{\|x\|} \leq Cond \times \frac{\|\Delta b\|}{\|b\|},$$

where

$$Cond = \frac{\sigma_{max}}{\sigma_{min}}.$$

——— Continued ———

【背面還有試題】

(b) To prove the bound,

$$\frac{\|\Delta x\|}{\|x\|} \leq Cond_{eff} \times \frac{\|\Delta b\|}{\|b\|},$$

where

$$Cond_{eff} = \frac{\|b\|}{\sigma_{min}\|x\|}.$$

In the above equations, $\|x\|$ is the 2- norm, and σ_{max} and σ_{min} are the maximal and the minimal singular values of matrix A, respectively.

——————————— end ———————————

國立中山大學97學年度博士班招生考試試題

科目：分析【應數系丙組選考】

Answer *all* of the following questions. Each carries 20 points.

1. (a) Prove that every convex function $f : [0, 1] \longrightarrow \mathbb{R}$ attains its maximum value at either 0 or 1.

   (b) Prove that every convex function $f : [0, 1] \longrightarrow \mathbb{R}$ is continuous on $(0, 1)$. What can you say about the continuity of f at the endpoints 0 and 1?

2. (a) Let f be a positive measurable function defined on $\mathbb{R}$. Show that there is a sequence $\{f_n\}_n$ of simple functions such that $f_n(x)$ monotonically increases to $f(x)$ everywhere.

   (b) Let f be a monotone (increasing or decreasing) real-valued function defined on $[a, b]$. Prove that f is continuous everywhere on $[a, b]$ except possibly for at most countably many points.

3. (Baire Category Theorem) Let X be a complete metric space and $\{X_n : n \in \mathbb{N}\}$ be a countable collection of closed subsets of X such that $X = \bigcup_n X_n$. Prove that at least one of X_n's has non-empty interior.

4. An extended real valued function $f : \mathbb{R} \longrightarrow [-\infty, +\infty]$ is said to be *lower semicontinuous* at the point y if $f(y) \neq -\infty$ and $f(y) \leq \liminf_{x \to y} f(x)$. Show the following statements.

   (a) Let $f(y)$ be finite. Then f is lower semicontinuous at y if and only if given $\epsilon > 0$, there is a $\delta > 0$ such that $f(y) \leq f(x) + \epsilon$ for all x with $|x - y| < \delta$.

   (b) A real valued function f is lower semicontinuous on (a, b) if and only if the set $\{x \in \mathbb{R} : f(x) > \lambda\}$ is open for each real number λ.

   (c) A lower semicontinuous real valued function f defined on $[a, b]$ bounded from below assumes its minimum on $[a, b]$.

   (d) (Dini Theorem) Let $\{f_n\}_n$ be a sequence of lower semicontinuous functions defined on $[a, b]$. Suppose $f_n(x)$ monotonically increasing to 0 for all x in $[a, b]$. Then f_n converges to zero uniformly on $[a, b]$.

【背面還有試題】

5. Let $I = [0,1]$ and $Q = I \times I$. Define $f : Q \longrightarrow \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{1}{p} & \text{if } y \text{ is rational and } x = \frac{q}{p}, p, q \in \mathbb{N} \text{ such that} \\ & \text{the greatest common factor } (p,q) \text{ of } p \text{ and } q \text{ is } 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Is f integrable over Q? If yes, compute $\int_Q f$.

(b) For each fixed x, compute the lower Riemann integral $\underline{\int}_{y\in I} f(x,y)$ and the upper Riemann integral $\overline{\int}_{y\in I} f(x,y)$.

(c) Show that $\int_{y\in I} f(x,y)$ exists for x in $I - D$, where D is a set of measure zero in I.

(d) Verify Fubini's theorem for $\int_Q f$.

End of Paper