國立中山大學九十四學年度博士班招生考試試題

科目：**機率論【應數系甲組】**

Do all problems in detail. 20 points for each problem. (06/02/2005)

(1) Let L^1 be the space of random variables with finite mean, and L^2 be the space of random variables with finite variance. Give and prove the inclusion relation between these spaces. Also, show that the inclusion is strict.

(2) State and prove the *Borel-Cantelli lemma.*

(3) Let $\{\mathrm{X}_n\}$, $n = 1, 2, \ldots$, be a sequence of random variables with finite second moments. Suppose that $\mathrm{X}_n \to \mathrm{X}$ in L^2, i.e., $E(|\mathrm{X}_n - \mathrm{X}|^2) \to 0$.

(i) Prove or disprove: $\mathrm{X}_n \to \mathrm{X}$ in probability.

(ii) Prove or disprove: The second moments of X_n converge to that of X.

(4) Let $\{\mathrm{X}_n\}$, $n = 1, 2, \ldots$, be a sequence of random variables that satisfies $\sqrt{n}(\mathrm{X}_n - \mu) \to N(0, \sigma^2)$ in distribution. Prove or disprove: $\mathrm{X}_n \to \mu$ in probability.

(5) Suppose that the conditional (given N) distribution of X is χ^2_{2N}, where the distribution of N is Poisson (θ). Calculate $E(\mathrm{X})$ and $Var(\mathrm{X})$.

End of Paper

國立中山大學九十四學年度博士班招生考試試題

科目：數理統計【應數系甲組】

共五題每題20分。答題時，每題都必須寫下題號與詳細步驟。

1. A cdf F_X is stochastically greater than a cdf F_Y if $F_X(t) \leq F_Y(t)$ for all t and $F_X(t) < F_Y(t)$ for some t. A family of cdfs $\{F(x|\theta), \theta \in \Theta\}$ is stochastically increasing in θ if $\theta_1 > \theta_2 \Rightarrow F(x|\theta_2)$ is stochastically greater than $F(x|\theta_1)$.

   (a) Prove that if $X \sim F_X(x|\theta)$, where the sample space of X is $(0, \infty)$ and $F_X(x|\theta)$ is stochastically increasing in θ, then $F_Y(y|\theta)$ is stochastically increasing in θ, where $Y = 1/X$.

   (b) Prove that if $X \sim F_X(x|\theta)$, where $F_X(x|\theta)$ is stochastically increasing in θ, and $\theta > 0$ then $F_X(x|\frac{1}{\theta})$ is stochastically increasing in θ.

2. If (X, Y) has the bivariate normal pdf

$$f(x,y) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left(\frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right),$$

   show that $\text{Corr}(X,Y) = \rho$ and $\text{Corr}(X^2, Y^2) = \rho^2$.

3. Suppose $X_1, X_2, \ldots$ are jointly continuous and independent, each distributed with marginal pdf $f(x)$, where each X_i represents annual rainfall at a given location.

   (a) Find the distribution of the number of years until the first year's rainfall, X_1, is exceeded for the first time.

   (b) Show that the mean number of years until X_1 is exceeded for the first is infinite.

4. Let $X_1, \ldots, X_{n+1}$ be iid Bernoulli(p), and define the function $h(p)$ by

$$h(p) = P\left(\sum_{i=1}^{n} X_i > X_{n+1} \middle| p\right),$$

   the probability that the first n observations exceed the $(n+1)$st.

   (a) Show that

$$T(X_1, \ldots, X_{n+1}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i > X_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

   is an unbiased estimator of $h(p)$.

   (b) Find the best unbiased estimator of $h(p)$.

5. Suppose that $X_1, \ldots, X_n$ are iid with a beta$(\mu, 1)$ pdf and $Y_1, \ldots, Y_m$ are iid with a beta$(\theta, 1)$ pdf. Also assume that the Xs are independent of the Ys.

   (a) Find an LRT of $H_0: \theta = \mu$ versus $H_1: \theta \neq \mu$.

   (b) Show that the test in part (a) can be based on the statistic

$$T = \frac{\sum \log X_i}{\sum \log X_i + \sum \log Y_i}.$$

   (c) Find the distribution of T when H_0 is true, and then show how to get a test of size $\alpha = .10$.

～全卷完～

國立中山大學九十四學年度博士班招生考試試題

科目：分析【應數系丙組】

(20%) 1. State and prove Baire's Theorem.

(20%) 2. If $A \subset [0, 2\pi]$ and A is measurable, prove that

$$\lim_{n\to\infty} \int_A \cos nx dx = \lim_{n\to\infty} \int_A \sin nx dx = 0.$$

(20%) 3. Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

(10%) 4. Let $a_1, a_2, ..., a_n$ be positive numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{1}{n}(a_1 + a_2 + \cdots + a_n).$$

(10%) 5. If X is compact and $f : X \to (-\infty, \infty)$ is upper semicontinuous, prove that f attains its maximum at some point of X.

(20%) 6. Let $\{f_n\}$ be a sequence of continuous complex functions on a nonempty complete metric space X such that $f(x) = \lim f_n(x)$ exists (as a complex number) for every $x \in X$.

(a) Prove that there is an open set $V \neq \emptyset$ and a number $M < \infty$ such that $|f_n(x)| < M$ for all $x \in V$ and for $n = 1, 2, 3, ...$

(b) If $\epsilon > 0$, prove that there is an open set $V \neq \emptyset$ and an interger N such that $|f(x) - f_n(x)| \leq \epsilon$ if $x \in V$ and $n \geq N$.

國立中山大學九十四學年度博士班招生考試試題

科目：組合數學【應數系丙組選考】

組合數學(Combinatorics)

June 2005

注意：每個問題必需證明或說明清楚。

1. Let $\left[{n \atop r}\right]$ be the set of all r-subsets of $\{1,2,\ldots,n\}$ and $K(n,r)$ be the Kneser graph with the vertex set $\left[{n \atop r}\right]$ and the edge set $\{AB: A,B \in \left[{n \atop r}\right]$ and $A \cap B = \phi\}$.

   (a) Prove that $K(n,r)$ is vertex transitive. (10%)

   (b) Prove that if $n \geq 2r$ then $K(n,r)$ is connected. (10%)

   (c) Is Petersen graph a Kneser graph? (10%)

2. A graph $G=(V,E)$ with n vertices is Hamiltonian-connected if every two vertices of G are connected by a Hamiltonian path. Prove that if $|E| \geq (n-1)(n-2)/2+3$ then G is Hamiltonian-connected. (20%)

3. Show that there exists an orientation D of a graph G such that the difference of indegree and outdegree of each vertex in D is 1, 0, or -1. (15%)

4. A graph G is self-complementary if it is isomorphic to its complement. Prove that if a graph G with n vertices is self-complementary then the chromatic number $\chi(G) \leq (n+1)/2$. (20%)

5. How many different integer solutions are there to the equation $x_1+x_2+x_3+x_4+x_5=30$ with $0 \leq x_i \leq 6$ for $i=1,2,3,4,5$? (15%)