國立中山大學九十二學年度博士班招生考試試題

共五題，每題20分。答題時，每題都必須寫下題號與詳細步驟。

1. Evaluate

$$\lim_{n\to\infty} \frac{1}{(n-1)!} \int_0^n x^{n-1} e^{-x}\, dx.$$

2. Let $X_1, \ldots, X_n$ be random variables, each being normally distributed with mean zero and variance 1. Find the maximum and minimum possible values of the variance of

$$Y = X_1 + \cdots + X_n$$

where $n \geq 2$ is fixed.

3. A die is rolled until all 6 faces have appeared at least once. Let X be the number of rolls needed. Find $E[X]$ and $E[X^2]$.

4. The function

$$\phi(t) = \frac{1}{(1+t^2)(1+2it)^2}$$

is a characteristic function of a probability density function $f(x)$. Find $f(x)$ for all x.

5. Consider a sequence of random numbers and let M denote the first one that is less than its predecessor. That is,

$$M = \min\{n : U_1 \leq U_2 \leq \cdots \leq U_{n-1} > U_n\}.$$

Find

(a) the probability mass function of M.

(b) the expectation of M.

～全卷完～

共五題，每題20分。答題時，每題都必須寫下題號與詳細步驟。

1. Show that if $(X, Y) \sim$ bivariate normal$(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$, then the following are true.

   (a) The marginal distribution of X is $N(\mu_X, \sigma_X^2)$ and the marginal distribution of Y is $N(\mu_Y, \sigma_Y^2)$.

   (b) The conditional distribution of Y given $X = x$ is
   $$N(\mu_Y + \rho(\sigma_Y/\sigma_X)(x - \mu_X), \sigma_X^2(1 - \rho^2)).$$

   (c) For any constant a and b, the distribution of $aX + bY$ is
   $$N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y).$$

2. Given that $N = n$, the conditional distribution of Y is χ_{2n}^2. The unconditional distribution of N is Poisson(θ).

   (a) Calculate $E[Y]$ and $\text{Var}(Y)$ (unconditional moments).

   (b) Show that, as $\theta \to \infty$, $(Y - E[Y])/\sqrt{\text{Var}(Y)} \to n(0, 1)$ in distribution.

3. Let $X_1, \ldots, X_n$ be iid with pdf
   $$f(x|\theta) = \theta x^{\theta - 1}, \quad 0 \le x \le 1, \quad 0 < \theta < \infty.$$

   (a) Find the MLE of θ, and show that its variance $\to 0$ as $n \to \infty$.

   (b) Find the method of moments estimator of θ.

4. Show that for a random sample $X_1, \ldots, X_n$ from a $\text{n}(0, \sigma^2)$ population, the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$, is given by
   $$\phi\left(\sum_{i=1}^n X_i^2\right) = \begin{cases} 1 & \text{if} \quad \sum_{i=1}^n X_i^2 > c \\ 0 & \text{if} \quad \sum_{i=1}^n X_i^2 \le c. \end{cases}$$
   For a given value of α, the size of the Type I Error, show how the value of c is explicitly determined.

5. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli(p).

   (a) Derive a confidence interval for p by inverting the likelihood ratio test of $H_0 :\ p = p_0$ versus $H_1 :\ p \neq p_0$.

   (b) Show that the interval is a highest density region from $p^y(1 - p)^{n-y}$.

～全卷完～

科目：分析【應數系丙組】

1. Let X be a campact Hausdorff space and $C(X)$ be the space of continuous functions on X. Prove that $C(X)$ is finite dimensional iff X is finite.

2. Let (X, d) be a metric space and A, B are subests of X. We say that A, B are *seperated* if $A \cap \text{cl}B = \text{cl}A \cap B = \emptyset$ (cl$S$=closure of S). A set $E \subseteq X$ is connected if E is not the union of two nonempty seperated sets.

a. Prove that disjoint open sets are seperated.

b. Fix $p \in X$, $\delta > 0$. Show that $A = \{x \in X : d(x, p) < \delta\}$ and $B = \{x \in X : d(x, p) > \delta\}$ are seperated.

c. Prove that every connected metric space with at least two points is uncountable (Hint: use (b)).

3. Let X be vector space over $\mathbb{R}$ and K is a convex subset of X. A subset E of K is called an *extreme subset* of K if (1) E is convex and nonempty and (2) if $x = ty + (1-t)z$ with $x \in E$, $y, z \in K$ and $t \geq 0$, then $y, z \in E$. A point $x \in K$ is an *extreme point* of K if $E = \{x\}$ is an extreme subset of K. Let ext(K) = The set of extreme points of K.

a. Show that (2) in the definition of extreme set can be replaced by the following statement:

$$\text{If } x = \frac{1}{2}y + \frac{1}{2}z \text{ with } x \in E,\ y, z \in K, \text{ then } y, z \in E.$$

b. Show that if E is an extreme subset of K and F is an extreme subset of E, then F is an extreme subset of K.

c. Let $l : X \longrightarrow \mathbb{R}$ be linear and $K_{\max}$ and $K_{\min}$ be subsets of K where l achieves its maximum and minimum on K, respectively. Show that, when nonempty, $K_{\max}$ and $K_{\min}$ are extreme subsets of K.

d. Use (c) to prove the Carathéodory theorem:

Let $K \neq \emptyset$ be a compact convex subset of $\mathbb{R}^n$, then ext$(K) \neq \emptyset$ and every point of K can be written as a convex combination of some $n+1$ extreme points of K (Hint: use induction on n).

4. What can you say about an entire function f whose range is inside the set $\{(x, y) : |x| + |y| > 1\}$?

5. Is $f(z) = e^z + e^{\sqrt{2}z}$ periodic ($z \in \mathbb{C}$)? Prove your answer. Warning: the period of f, if exists, may not be real.

6. Describe the set

$$2\left|z - \frac{1}{2}\right| \leq \left|1 - \frac{1}{2}z\right|,\ z \in \mathbb{C}.$$

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科目：組合數學【應數系丙組】

1 (17 points). Let G be a graph with mk edges. Prove that if G is k-edge colorable, then there is a k-edge coloring f of G in which every color class contains exactly m edges.

2 (17 points). A *homomorphism* from a digraph G to a digraph H is a mapping $f : V(G) \to V(H)$ such that $\overrightarrow{f(x)f(y)}$ is an arc of H whenever $\overrightarrow{xy}$ is an arc of G. Let $\overrightarrow{T_n}$ be the transitive tournament on n vertices, i.e., $\overrightarrow{T_n}$ has vertices $x_0, x_1, \cdots, x_{n-1}$, in which $\overrightarrow{x_i x_j}$ is an arc if and only if $i < j$. A directed walk of length n in a digraph G is a sequence $v_0, v_1, \cdots, v_n$ of (not necessarily distinct) vertices of G such that $\overrightarrow{v_i v_{i+1}}$ is an arc for $i = 0, 1, \cdots, n-1$. Prove that there is a homomorphism from a digraph G to $\overrightarrow{T_n}$ if and only if G has no directed walk of length n.

3 (16 points). Prove that if a cubic graph G has a Hamilton cycle, then G is 3-edge colorable.

4 (16 points). Prove that an r-regular bipartite graph can be partitioned into k-regular bipartite subgraphs if and only if k is a factor of r.

5 (17 points). Prove that a graph G is a forest if and only if every pairwise intersecting family of paths in G has a common vertex.

6 (17 points). The Ramsey number $R(p, q)$ is the least integer n for which the following holds: If the edges of K_n are colored by two colors, say red and blue, then there is either a red K_p (i.e., a K_p as a subgraph of K_n all of its edges are colored red) or a blue K_q. Prove that $R(p, q) \leq \binom{p+q-2}{p-1}$.