N.S.Y.S.U. Ph.D Entrance Exam for Applied Math (probability) 89.6.17.

25 points for each of the following problems

1. Let X and $\{X_n\}_{n=1}^{\infty}$ be random variables. Consider the following three statements:
(i) X_n converges to X in distribution as n tends to ∞.
(ii) X_n converges to X in probability as n tends to ∞.
(iii) X_n converges to X almost sure as n tends to ∞.

Answer the following questions:
(a) Prove that (iii)implies (ii).
(b) Prove that (ii) implies (i).
(c) Give a counter example that (ii) does not imply (iii).
(d) Give a counter example that (i) does not imply (ii).

2. Suppose that $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space, $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$.
(a) prove that $P(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty} P(A_n) \le \limsup_{n\to\infty} P(A_n)$ $\le P(\limsup_{n\to\infty} A_n)$.
(b) Find an example of $\{A_n\}_{n=1}^{\infty}$ so that $P(\liminf_{n\to\infty} A_n) = 1/3$ but $\liminf_{n\to\infty} P(A_n)$ $= 2/3$.
(c) Show that if the probability measure P is replaced by an arbitrary measure U then $U(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty} U(A_n)$ is still true.
(d) Give a counter example that $\limsup_{n\to\infty} U(A_n) \le U(\limsup_{n\to\infty} A_n)$ is not necessary true.

3. (a) Suppose $X_1, X_2, ..., X_n$ are i.i.d. random variables with mean μ (without loss of generality we may assume that $\mu = 0$) and finite fourth moment. Let $S_n = \sum_1^n X_i$. Prove the strong law of large number that S_n/n converges to 0 almost sure.
(b) If we only assume that $E(X_1^2) < \infty$ in (a) then the strong law of large number is also true but the proof is more complicated. Follow the follwing steps to complete the proof:
(i) Consider the sequence $\frac{S_1}{1}, \frac{S_4}{4}, ..., \frac{S_{n^2}}{n^2}, ...$Use Chebyshev's inequality and Borel-Cantelli Lemma to show that S_{n^2}/n^2 tends to 0 almost sure when $n \to \infty$.
(ii) Consider the distance function D_n defined by

$$D_n = max_{n^2 \le k < (n+1)^2} |S_k - S_{n^2}|. \quad (1)$$

Prove that

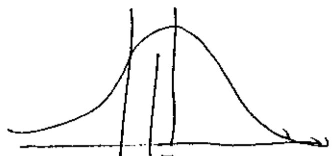
$$E(D_n^2) \leq ((n+1)^2 - n^2)E(X_1^2). \qquad (2)$$

(iii) Use Chebyshev's inequality and Borel-Cantelli Lemma to show that D_n/n^2 tends to 0 almost sure as $n \to \infty$.
(iv) Express $|S_k/k|$ as

$$|\frac{S_k}{k}| \leq \frac{S_{n^2} + D_n}{n^2} \qquad (3)$$

$\forall\, n^2 \leq k < (n+1)^2$. Prove that S_k/k tends to 0 almost sure as $k \to \infty$.

4. (a) Suppose that $(\Omega, \mathcal{F}, \text{u})$ is an arbitrary measure space and f, $\{f_n\}_{n=1}^{\infty}$ are measurable functions defined on $(\Omega, \mathcal{F}, \text{u})$.
State any 3 theorems (or results that you know) such that $\lim_{n\to\infty} f_n = f$ a.s. implies $\lim_{n\to\infty} \int f_n du = \int f du$.
(b) Prove any one of the theorems (or the results).



N.S.Y.S.U. Ph.D Entrance Exam for Applied Math (statistics) 89.6.17.

There are 5 problems and each problem has its scores

1. 5% (a) State the Central Limit Theorem for the i.i.d. case.
10% (b) Suppose that $X_1, X_2, ..., X_{15}$ are i.i.d. Poisson $P(\lambda)$ random variables with mean $\lambda = 3$. Find $P\{\sum_1^{15} X_i \geq 40\}$.
10% (c) To study whether to build the fourth nuclear power plant in Taiwan, a survey is needed. Suppose a sample $X_1, X_2, ..., X_n$ of size n is collected where each X_i is binomial $b(1, p)$ and the parameter p is unknown. $\bar{X}$ is the best estimator of p. What is the minimum size of n if more than 95 % confidence is needed to guarantee that the estimation is within 3% (it means: $P\{|\bar{X} - p| \leq 0.03\} \geq 0.95$).

2. Suppose the random variable X is binomial $b(7, p)$, p belongs to $\Omega = \{1/2, 2/3, 3/4\}$. Answer the following questions:
5% (a) If we observe $X = 5$. What is the maximum likelihood estimate of p.
10% (b) To test $H_0 : p = 1/2$ against $H_1 : p = 2/3$ or $3/4$ with type I error $\alpha = 0.10$. If we observe $X = 5$, should we reject H_0 or accept H_0?

3. Suppose that $X_1, X_2, ..., X_n$ are i.i.d. normal $N(\mu, 1)$. To estimate μ^2.
10 % (a) Prove that the Cramer-Rao lower bound for any unbiased estimator is $4\mu^2/n$.
10 % (b) Prove that $T(X) = \bar{X}^2 - \frac{1}{n}$ is UMVUE (uniformly minimum variance unbiased estimator) for μ^2. Find the variance $Var(T(X))$.

4. Suppose that X_1, X_2 are i.i.d. normal $N(0, 1)$. Answer the following questions:
10 % (a) Find the distribution of the random variable X_1/X_2.
10 % (b) Let $Y_1 = X_1^2 + X_2^2$, $Y_2 = X_1/\sqrt{Y_1}$. Prove that Y_1 and Y_2 are independent.

5. Suppose that $X_1, X_2, ..., X_n$ are i.i.d. r.v's with distribution $F(x)$ and $Y_1, Y_2, ..., Y_m$ are i.i.d. r.v's with distribution $G(y)$. To test the hypothesis $H_0 : F(x) = G(x)$ for all x against the alternative $H_1 : F(x) \neq G(x)$ for at least one x. Suppose we use Wilcoxon rank test that we give rank k to the k-th order statistic in the combined samples.
10 % (a) Let W be the total rank for the Y samples. Find the mean and variance of W under H_0.

10 % (b) Suppose in an international software design competition, Taiwan sends 12 competitors and the scores are 46,88,79,95,82,92,100,49,64,68,77,85. Mainland China also sends 12 competitors and the scores are 74,58,62,46,55,60,88, 90,45, 57,60,61. Test if the performance of Taiwanese competitors is superior to that of Mainland competitors. (type I error is 0.05).

國立中山大學八十九學年度碩博士班招生考試試題

科　目：應數所乙組（數值分析）　　共1頁 第1頁

每題20分，無計算或證明過程者，不予計分。禁止使用計算器。

1. Let $f \in C^{2n+1}[a,b]$ and $x_1, x_2, \cdots, x_n \in [a,b]$ distinct.

   ① Find basis functions $u_i(x)$ and $v_i(x)$ of degree $\le 2n-1$ such that
   $\begin{cases} u_i(x_j) = \delta_{ij} \\ u_i'(x_j) = 0 \end{cases}$ and $\begin{cases} v_i(x_j) = 0 \\ v_i'(x_j) = \delta_{ij} \end{cases}$, $\forall i, j = 1, \cdots, n$. $\left(\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \ne j \end{cases}\right)$

   ② Show that $\exists !$ polynomial $H(x)$ of degree $\le 2n-1$ such that $H(x_i) = f(x_i)$ and $H'(x_i) = f'(x_i)$, $\forall i = 1, \cdots, n$.

   ③ What's the error $H(x) - f(x)$?

2. Let $g \in C[a,b]$ and $g([a,b]) \subset [a,b]$, show that ① g has at least one fixed point in $[a,b]$. Futhermore, assume $\exists\, \lambda \in (0,1)$ s.t. $|g(x) - g(y)| \le \lambda |x-y|$ $\forall x, y \in [a,b]$, show that ② g has a unique fixed point $\alpha \in [a,b]$,

   ③ $\forall x_0 \in [a,b]$, $x_{n+1} = g(x_n)$ converges to α (as $n \to \infty$) linearly,

   ④ $|x_n - \alpha| \le \lambda^n \max\{x_0 - a,\ b - x_0\}$, ⑤ $|x_n - \alpha| \le \frac{\lambda^n}{1-\lambda} |x_1 - x_0|$,

   ⑥ $|x_n - \alpha| \le \frac{\lambda}{1-\lambda} |x_n - x_{n-1}|$, $\forall n \in \mathbb{N}$.

3. Let $A \in \mathbb{R}^{n \times n}$ be diagonalizable with eigenvalues $\lambda_1, \cdots, \lambda_n$.

   ① Assume $|\lambda_1| > |\lambda_2| \ge \cdots \ge |\lambda_n|$, state the power method to compute λ_1 and its eigenvector.

   ② How to choose initial vector (in theory and in computation) so that the power method converges? Prove convergence under your assumption.

4. ① State Jacobi and Gauss-Seidel methods to solve linear system.

   ② For strictly diagonally dominant matrix A, use maximum row sum matrix norm $\|\cdot\|_\infty$ to show Gauss-Seidel method for $A\mathbf{x} = \mathbf{b}$ converges faster than Jacobi method.

5. State how to use ① normal equation, ② QR factorization ③ singular value decomposition to solve least square problem $A\mathbf{x} = \mathbf{b}$, where $m \ge n$, $A \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$. Sketch the proofs of above methods.

國立中山大學八十九學年度碩博士班招生考試試題

科　目：應數所丙組（分析數學）

Answer *all* of the following questions.

1. Let f be monotone (increasing or decreasing) on $[a, b]$. Prove that f is integrable on $[a, b]$.

2. Prove that if f is a real function on a measurable space X such that $\{x \in \mathbb{R} : f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

3. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

   (a) Define the notion of *bounded linear functionals* of $L^p[0, 1]$.

   (b) Prove that if $g \in L^q[0, 1]$ then the linear functional

   $$M(f) := \int fg\, dm, \quad f \in L^p[0, 1].$$

   is bounded, and compute its bound $\|M\|$.

   (c) State, without proof, Riesz representation theorem for bounded linear functionals of $L^p[0, 1]$.

4. Let f be a real-valued measurable function on a measure space $(X, \mathcal{A}, \mu)$. Show that

   (a) If $\mathcal{B}$ is the class of sets B in $\mathbb{R}$ such that $f^{-1}(B) \in \mathcal{A}$ then $\mathcal{B}$ is a σ–algebra which contains the Borel sets.

   (b) If $\nu(B) := \mu(f^{-1}(B))$ for all B in $\mathcal{B}$ then ν is a measure on $\mathcal{B}$.

5. Let X be a complete metric space and $\{O_n : n \in \mathbb{N}\}$ a countable collection of dense open subsets of X. Prove that $\bigcap_n O_n$ is dense in X.

End of Paper

1. (25points) Suppose G is a connected graph. Let $\alpha'(G)$ denote the size of a maximum matching of G, and $\beta(G)$ denote the size of a minimum vertex covering of G. Prove that

- $2\alpha'(G) \geq \beta(G)$.
- If G is bipartite then $\alpha'(G) = \beta(G)$.

2. (15 points) Let X be an n-element set. A mapping $f : A \to A$ is called *fix point free* if for every $x \in A$, $f(x) \neq x$. How many fix point free one-to-one mappings are there ?

3. (15 points) Prove a connected graph with $n \geq 2$ vertices has at least two vertices which are not cut-vertices.

4. (15 points) Prove that if a graph G has minimum degree $\delta(G) \geq |V(G)|/2$, then G has a Hamilton cycle.

5. (15 points) Let $G = (V, E)$ be a graph, and let $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ be a decomposition of G, i.e., G_1, G_2 are two subgraphs of G, where $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E$. Prove $\chi(G) \leq \chi(G_1) \times \chi(G_2)$.

6. (15 points) Prove that a graph is bipartite if and only if G contains no odd cycles.