共四大題 每大題 25分

一、我們定義 $\overline{\lim} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$。今投擲一正常之硬幣，$P(\text{正面}) = P(\text{反面}) = \frac{1}{2}$。

6 (a) 設 $A_n = \{$第 n 次投擲出現為正面$\}$，求 $P(\overline{\lim} A_n)$。

6 (b) 設 $A_1 = \{$前 3 次投擲中有二次正面$\}$

$A_2 = \{$接下來的 6 次投擲中有二次正面$\}$

$A_3 = \{$接下來的 9 次投擲中有二次正面$\}$

$\vdots$

求 $P(\overline{\lim} A_n)$。

6 (c) 設 $(\Omega, \mathcal{F}, P)$ 為任一 probability space，$\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$

證明 $\overline{\lim} P(A_n) \leq P(\overline{\lim} A_n)$ 一定成立。

7 (d) 但若 $(\Omega, \mathcal{F}, \mu)$ 為任一 measure space，$\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$

則 $\overline{\lim} \mu(A_n) \leq \mu(\overline{\lim} A_n)$ 卻不一定成立。

試舉一例說明之。

二、設 $\{X_n\}_{n=1}^{\infty}$ 為 $(\Omega, \mathcal{F}, P)$ 上的 uniformly bounded r.v's for some $M > 0$。

即 $|X_n| \leq M \quad \forall n = 1, 2, \cdots$

(a) 證明：若 $\lim_{n\to\infty} X_n(\omega) = X(\omega) \ \forall \omega \in \Omega$，則 $E(X_n) \to E(X)$。

(b) 舉一反例說明若沒有 uniformly bounded 的條件則 (a) 不一定正確。

三. (a) We say that f is integrable w.r.t the measure μ. if $\int f^{+} d\mu < \infty$ and $\int f^{-} d\mu < \infty$. prove that if f is integrable, then for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that $\mu(A) < \delta \Rightarrow \int_A |f| d\mu < \varepsilon$.

(b) We say that $\{f_n\}_{n=1}^{\infty}$ are uniformly integrable if $\exists\ M > 0$ s.t $\int_{|f_n| \geq M} |f_n| d\mu < \varepsilon$, (ε is given)

prove that if $\{f_n\}_{n=1}^{\infty}$ are uniformly integrable, $f_n \geq 0$ and let $g_n = \max_{1 \leq i \leq n} \{f_i\}$. Then $\int_{g_n \geq \alpha} g_n d\mu \leq \sum_{k=1}^{n} \int_{f_k \geq \alpha} f_k d\mu$

四 (a) 試敘述 Central Limit Theorem 中的 Lindeberg Condition

(b) Let $P\{X_{nk} = 1\} = \frac{1}{n-k+1} = 1 - P\{X_{nk} = 0\}$. $k = 1, 2, \cdots, n$

where for each n, X_{nk} are independent;

Show that if $S_n = \sum_{k=1}^{n} X_{nk}$, then

$$\frac{S_n - \log n}{\sqrt{\log n}} \xrightarrow{\mathcal{D}} N(0,1).$$

國立中山大學八十七學年度碩博士班招生考試試題

科　目：數理統計　　　　應用數學系　共 / 頁　第 / 頁

1. Let X and Y be independent random variables with density function

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the cumulative distribution functions of X/Y and XY. (15%)

2. Let $x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_n > 0$ and $\sum_{i=1}^n w_i = 1$. Show that

   (a) $\frac{1}{\sum_{i=1}^n w_i x_i} < \sum_{i=1}^n w_i \left(\frac{1}{x_i}\right)$. (10%)

   (b) $\ln\left(\sum_{i=1}^n w_i x_i\right) > \sum_{i=1}^n w_i \ln x_i$. (10%)

3. Let Y be normally distributed with mean $g(\beta_1, \beta_2) = \beta_1 x/(\beta_2 + x)$ and variance 1 where β_i's are interested unknown parameters and x is a fixed constant. Find the Fisher information matrix for (β_1, β_2). (15%)

4. Let (X_i, Y_i), $i = 1, 2, \ldots, n$ be independent bivariate normal distributed with mean (μ_x, μ_y) and covariance matrix Σ where μ_x, μ_y and Σ are unknown parameters.

   (a) Show that $g = \sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})/(n-1)$ is an unbiased estimator of Σ. (10%)

   (b) Is the estimator g a uniformly minimum variance unbiased estimators? (10%)

5. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from $U(\theta, 2\theta)$, $\theta \in (0, \infty)$, distribution and set

$$U_1 = \frac{n+1}{2n+1} X_{(n)} \quad \text{and} \quad U_2 = \frac{n+1}{5n+4}[2X_{(n)} + X_{(1)}]$$

where $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ are the order statistic of $X_1, X_2, \ldots, X_n$. Show that both U_1 and U_2 are unbiased estimators of θ and that U_2 is uniformly better than U_1 in the sense of variance. (20%)

6. Let $X_1, X_2, \ldots, X_n$ denote the incomes of n persons chosen at random from a certain population. Suppose that each X_i has the Pareto density

$$f(x, \theta) = c^\theta \theta x^{-(1+\theta)}, \quad x > c$$

where $\theta > 1$ and $c > 0$. Find the optimal test statistic for testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu > \mu_0$ where $\mu = E(X_i)$. (10%)

So faith comes out of hearing, and hearing through the word of Christ.

– Roman 10:17

國立中山大學八十七學年度[crossed out]博士班招生考試試題

1. (15%) Let G be a weighted undirected graph, with vertex set $\{1, 2, \ldots, n\}$. For $i = 1, 2, \ldots, n$, let e_i be the edge with minimum weight among all edges incident with vertex i.

   (a) For each $i = 1, 2, \ldots, n$, show that there is a minimum spanning tree that contains the edge e_i.

   (b) Let $A = \cup_{i=1}^{n} e_i$. Show that there is a minimum spanning tree that contains all the edges in A.

   (c) Is the induced graph $G[A]$ a minimum spanning tree of G? If it is not, show how to construct a minimum spanning tree of G efficiently, from the set of edges in A.

2. (10%) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be $2n$ distinct numbers such that $a_i < b_i$ for $i = 1, 2, \ldots, n$. Suppose that $a_1, a_2, \ldots, a_n$ are rearranged in decreasing order as $a'_1, a'_2, \ldots, a'_n$, and $b_1, b_2, \ldots, b_n$ are rearranged in decreasing order as $b'_1, b'_2, \ldots, b'_n$. Show that $a'_i < b'_i$ for $i = 1, 2, \ldots, n$.

3. (15%) Solve the following recurrence equations for $n > 0$.

   (a) $T(n) = \begin{cases} 0, & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + \log n, & \text{if } n > 1. \end{cases}$

   (b) $U(n) = \begin{cases} 1, & \text{if } n = 1, 2 \\ U(n-1) + 2U(n-2) + (-1)^n, & \text{if } n > 2. \end{cases}$

   (c) $V(n) = \begin{cases} 1, & \text{if } n = 1 \\ V(n-1) + 2V(n-2) + \ldots + (n-1)V(1) + n, & \text{if } n > 1. \end{cases}$

4. (20%) Let K_n be a complete graph with n vertices. The edges of K_n are to be colored in either red or blue. A subgraph K_i of K_n is call red if all edges of K_i are colored red. A subgraph K_i is call blue if all edges of K_i are colored blue.

   (a) Show that, in any coloring of K_6, there is a subgraph which is a red K_3 or a blue K_3.

   (b) Show that if there are 6 red edges incident with one vertex then there is either a subgraph which is a red K_4 or a blue K_3.

   (c) Show that if there are 4 blue edges incident with one vertex then there is a subgraph which is a red K_4 or a blue K_3.

   (d) Using the above results, show that any coloring of K_9 contains a subgraph which is a red K_4 or a blue K_3.

5. (10%) Define the *regular set*. Show that the intersection of two regular sets is a regular set.

6. (10%) Design a context-free grammar for the language of binary strings with equal number of 0's and 1's. Briefly explain why your grammar generates the language.

7. (20%) Define the complexity classes P, NP, co-P, co-NP, and NP-complete.
   Prove each of the following statements.

   (a) P = co-P.

   (b) NP $\neq$ co-NP implies P $\neq$ NP

   (c) If there is an NP-complete problem A such that $A \in$ P then P = NP.

   (d) If there is an NP-complete problem A such that $A^c \in$ NP then NP = co-NP, where A^c is the complement of A.

1. (10%)
Suppose that a moving-head disk contains 200 tracks (numbered 0 through 199) and that the head is currently serving a request at track 143 and has just finished a request at track 126. We have the following requests in the FIFO order:
86, 147, 91, 177, 94, 150, 102, 175, 130.
What is the total number of head movment needed to satisfy these requests for the following disk-scheduling algorithms ?
(a) FCFS.
(b) SSTF.
(c) SCAN.
(d) LOOK.
(e) C-SCAN.

2. (10%)
Assume we have a paged memory system with associative registers to hold the most active page table entries. If the page table is normally held in memory, and memory access time is 1 microsecond, what is the effective access time if 85% of all memory references find their entries in the associative registers ? How about 50% ?

3. (10%)
(a) In the following two-process solution for mutual exclusion, what is wrong with it ?

```
repeat
  flag[i]:=true;  --(1)
  while flag[j] do no-op;  --(2)
  CS;
  flag[i] :=false;
until false;
```

(2) If we exchange the order of statements 1 and 2, will the algorithm be correct ?

4. (10%)
Consider a file currently consisting of 100 blocks. Assume that the file control block is already in memory. Calculate how many disk I/O operations are required for contiguous and linked allocation strategies, if, for one block, the following condition hold. In the contiguous-allocation case, assume that there is no room to grow in the beginning, but there is room to grow in the end. Assume that the block information to be added is stored in memory.
(a) The block is added at the beginning.
(b) The block is added in the middle.
(c) The block is added at the end.
(d) The block is removed form the beginning.
(e) The block is removed from the middle.

5. (10%)
Given memory partitions of 100K, 500K, 200K, 300K and 600K (in order), how would each of the First-fit and Best-fit algorithms place processes of 212K, 417K, 112K and 426K (in order) ?

6. (10%)
Show that the exclusive-OR function x = A ++ B ++ C ++ D is an odd function, where ++ denotes the exclusive-OR operator. One way to show this is to obtain the truth table for x = y ++ z. Show that x = 1 only when the total number of 1's in A, B, C and D is odd.

7. (10%)
A sequential circuit has two D flip-flops A and B, two inputs x and y, and one output z. The flip-flop input equations and the circuit output are as follows:

$D_A = x'y + xA$

$D_B = x'B + xA$

$z = B$

(a) Draw the logic diagram of the circuit.
(b) Tabulate the state table.

8. (10%)
Write a BNF (or EBNF) grammar which can accept the expression A + B * C **D - E/D, where ** is the exponential operator.

9.(10%)

Suppose that a grammar has the following productions:

S-->aBc

B-->bXb

B-->bX

X-->a

X-->ab

(a) Is the grammar ambiguous ? Justify your answer.

(b) Describe the set of all strings in the language which is generaged by the grammar.

10. (10%)

Describe 5 types of interrupts; moreover, for each type, describe two examples.

國立中山大學八十七學年度碩博士班招生考試試題

科 目：分析　　　　應用數學系　　共 1 頁 第 1 頁

1. Find the integral of $f(x,y,z) = x - z$ over the region bounded by $z = y^2$, $z = 1$, $z = x$ and $x = 0$. (10%)

2. Suppose V is an open subset of $\mathbb{R}^2$, $H = [a,b] \times [0,c] \subset V$, $u : V \to \mathbb{R}$ is C^2 on V, and $u(x_0, t_0) \geq 0$ for all $(x_0, t_0) \in \partial H$. ($\partial H$ is the boundary of H.)

   (a) Show that given $\varepsilon > 0$, there is a compact set $K \subset H^\circ$ (H° is the interior of H.) such that $u(x,t) \geq -\varepsilon$ for all $(x,t) \in H \setminus K$. (10%)

   (b) Suppose $u(x_1, t_1) = -\ell < 0$ for some $(x_1, t_1) \in H^\circ$ and choose $r > 0$ so small that $2rt_1 < \ell$. Apply part (a) to $\varepsilon = \frac{\ell}{2} - rt_1$ to choose the compact set K, and prove that the minimum of
   $$w(x,t) = u(x,t) + r(t - t_1)$$
   on H occurs at some $(x_2, t_2) \in K$. (10%)

   (c) Prove that if u satisfies the heat equation, i.e. $u_{xx} - u_t = 0$ on V, and if $u(x_0, t_0) \geq 0$ for all $(x_0, t_0) \in \partial H$, then $u(x,t) \geq 0$ for all $(x,t) \in H$. (10%)

3. Suppose f is analytic on $(-\infty, \infty)$ and
$$\int_a^b |f(x)|\,dx = 0$$
for some $a \neq b$ in $\mathbb{R}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$. (20%)

4. Show that any uncountable subset A of a second countable space must have at least one point which is a limit point of A. (20%)

5. Let N be a non-empty normed linear space. Prove that N is a Banach space if and only if the set $\{x : \|x\| = 1\}$ is complete. (20%)

國立中山大學八十七學年度碩博士班招生考試試題

1 (14 points). Suppose G is a connected 7-regular graph (i.e., each vertex has degree 7). Prove that G has no cut-edge.

2 (15 points). Prove that for any graph G, $\chi(G)\chi(\bar{G}) \geq n$, where $\bar{G}$ is the complement of G, and n is the number of vertices of G.

3 (14 points). How many bipartite subgraphs of K_n are needed to cover all the edges of K_n ? Why ?

4 (14 points). Prove that every planar graph is 5-colorable.

5 (14 points). Prove that if G is a 3-regular graph which has a Hamiltonian cycle, then G is 3-edge colorable.

6 (15 points). Suppose $G = (V, E)$, $x, y \in V$ and $d_G(x) + d_G(y) \geq |V|$. Prove that G has a Hamiltonian cycle if and only if $G + xy$ has a Hamiltonian cycle. Here $G + xy$ denotes the graph obtained from G by adding the edge xy.

7 (14 points). Describe an algorithm that finds a maximum matching in a bipartite graph.

Entrance Exam for the Ph.D Program of Scientific Computing

Seven questions with the marks indicated.

1.(15) Let

$$A \in R^{n\times n},\ Cond.(A) = \{\frac{\lambda_{max}(A^TA)}{\lambda_{min}(A^TA)}\}^{1/2},$$

where $\lambda_{max}(A)$ and $\lambda_{min}(A)$ are the maximal and minimal eigenvalues of matrix A respectively. Prove
(1). $Cond.(AB) \le Cond.(A)Cond.(B)$,
(2) $Cond.(UA) = Cond(A)$, where $U \in R^{n\times n}$ is an orthogonal matrix.

2.(15) Suppose that there exists a root of $f(x) = 0$, and $0 \le m \le f'(x) \le M$. Prove that

$$x_{n+1} = x_n - \lambda f(x_n)$$

yields the convergent sequence $\{x_n\}$ to the root for arbitrary $x_0 \in (-\infty, \infty)$ and $0 < \lambda < 2/M$.

3.(10) Let $Ax = b,\ A\tilde{x} = \tilde{b}$, where $|A| \neq 0$. Prove

$$\frac{||x - \tilde{x}||}{||x||} \le \|A\|\|A^{-1}\|\frac{\|b - \tilde{b}\|}{\|b\|}$$

where $x \neq 0$, $b \neq 0$ and $\|x\|$ is any vector norm.

4.(15) Solve the overdetermined system

$$Ax = b,\ A \in R^{m\times n},\ x \in R^n\ ,b \in R^m,\ m > n,$$

with $Rank(A) = n$. Give a solution method and the corresponding stability analysis.

5.(15) Give the composite central rule for evaluating the two dimensional integral $\int_0^1\int_0^1 f(x,y)dxdy$, and derive its error bounds.

6.(15) Consider the initial value problem of ODE,

$$y' = f(x,y), x \ge a;\ y(a) = y_0.$$

Give the trapezoidal scheme, derive the local truncation errors and provide stability analysis.

7.(15) Write down the five point-difference scheme for solving

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial x^2} = f(x,y)\ \ in\ \ S, \quad u = g(x,y)\ \ on\ \ \partial S,$$

with the unit square $S : \{0 \le x \le 1,\ 0 \le y \le 1\}$. Provide the relaxation iteration method and describe in detail the optimal relaxation parameter.