Do all problems in details.

1. (20%) Find $P(X > 2Y)$ if X and Y are jointly distributed with pdf $f(x,y) = x + y$, $0 \le x \le 1, 0 \le y \le 1$.

2. (20%) Let $X_1, \ldots, X_n$ be a random sample from the probability mass function $p(x)$ with an unknown parameter θ:

$$p(x) = \begin{cases} \theta^2, & \text{if} \quad x = 1; \\ (1-\theta)^2, & \text{if} \quad x = 2; \\ 2\theta(1-\theta), & \text{if} \quad x = 3. \end{cases}$$

(a) Find the maximum likelihood estimator (MLE) of θ.

(b) Calculate the observed Fisher information at MLE.

3. (15%) The survival time (day) is collected for 8 patients with advanced lung cancer as listed below:

$$655, 203^+, 92, 221^+, 11, 293, 167, 284^+.$$

The plus sign indicates the patient still alive at the end of the study. Physicians wished to determine if the median survival time was longer than 180 days. Perform an appropriate hypothesis testing at 5% level of significance and draw conclusion accordingly.

4. (15%) Find a sufficient statistic for μ if a random sample $X_1, \ldots, X_n$ coming from a distribution with density $f(x) = \dfrac{1}{2}\exp^{-|x-\mu|}$, $x \in \mathbb{R}$. Is the statistic complete for this distribution or not?

5. (20%) Let $\{X_n\}$ be independent Bernoulli(p). Set $\bar{X}_n = \sum_{i=1}^n X_i/n$. Show that

$$\sqrt{n}(\bar{X}_n - p) \to \text{normal}(0, p(1-p)) \quad \text{in distribution.}$$

6. (10%) Which one (and only one) do you think is the most important concept in statistical inference? Describe that concept intuitively, i.e., using as few formulas as possible. Also explain why the concept is so important for model validation, decision making, etc.

End of paper

國立中山大學108學年度博士班招生考試試題

科目： 分析 【應用數學系博士班丙組】

本科目依簡章規定不可使用電子計算機。

共七道題。答題時，每題須寫下題號與詳細步驟。

1. [14%] Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that both limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ exist and are finite. Prove that f is uniformly continuous on $\mathbb{R}$.

2. [14%] Let f be a real-valued differentiable function on $[-2, 2]$ satisfying $f'(-1) < f'(1)$. Show that for any number c in the open interval $(f'(-1), f'(1))$, there exists a pint $x_0 \in (-1, 1)$ for which $f'(x_0) = c$.

3. [14%] Given a non-negative and continuous function f on $[0, 1]$, define f_n on $[0, 1]$ iteratively by $f_0 = f$ and
$$f_{n+1}(x) = \int_0^x f_n(t)\, dt, \qquad n = 0, 1, \ldots.$$
Show that $f_n \to 0$ uniformly on $[0, 1]$.

4. [14%] Let $\{a_n\}$ be a positive sequence with $\sum a_n$ divergent. Prove that the series
$$\sum \frac{a_n}{1 + a_n}$$
also diverges.

5. [14%] Prove that the equation
$$x^2 + x + y + \sin(x^2 + y^2) = 0$$
determines a unique solution y as a function x near the point $(0, 0)$ and that this unique solution is differentiable at 0. Find the derivative $y'(0)$.

6. [15%] Let f and g be both real-valued, Lebesgue integrable, and non-increasing functions over $[0, 1]$. Show that
$$\int_0^1 fg \geq \int_0^1 f \cdot \int_0^1 g.$$

7. [15%] Suppose that $f : [0, 1] \to \mathbb{R}$ is Lebesgue integrable and satisfies
$$\int_0^1 f(x)x^n\, dx = 0, \qquad n = 0, 1, 2, \ldots.$$
Prove that $f = 0$ a.e. on $[0, 1]$.

End of Paper