每題 20 分，共 5 題，請詳列計算和推導過程書寫於題目下方空白處。

問題	1: 20 分	2: 20 分	3: 20 分	4: 20 分	5: 20 分	總分: 100 分
得分						

1. Let $(X_1, \ldots, X_n)$ be a random sample from the uniform distribution on the interval $[0, 1]$ and let $R = X_{(n)} - X_{(1)}$, where $X_{(i)}$ is the ith order statistic. Derive the probability density function of R and find the limiting distribution of $2n(1 - R)$.

2. Let $(X_1, \ldots, X_n)$ be a random sample from the uniform distribution on the interval $\left(\theta - \frac{1}{2}, \theta + \frac{1}{2}\right)$, where $\theta \in \mathbb{R}$ is unknown. Let $X_{(j)}$ be the jth order statistic. Show that $(X_{(1)} + X_{(n)})/2$ is strongly consistent for θ and also consistent in mean squared error.

3. Let $(X_1, \ldots, X_n)$ be a random sample from a population having the probability density function

$$f_{\theta_1,\theta_2}(x) = \begin{cases} (\theta_1+\theta_2)^{-1}e^{-x/\theta_1} & x > 0 \\ (\theta_1+\theta_2)^{-1}e^{x/\theta_2} & x \le 0 \end{cases}$$

where $\theta_1 > 0$ and $\theta_2 > 0$ are unknown. Obtain a moment estimator of (θ_1, θ_2) and its asymptotic distribution.

4. Let $(x_1, \ldots, x_n)$ be a random sample from a population on $\mathbb{R}$ with probability density function f_θ. Let θ_0 and θ_1 be two constants. Find a UMP test of size α for testing $H_0\colon \theta = \theta_0$ versus $H_1\colon \theta = \theta_1$ in the following cases:

   (i) $f_\theta = e^{-(x-\theta)} I_{(\theta,\infty)}(x)$, $\theta_0 < \theta_1$;

   (ii) $f_\theta(x) = \theta x^{-2} I_{(\theta,\infty)}(x)$, $\theta_0 \neq \theta_1$.

5. Let $X = (X_1, \ldots, X_n)$ be a random sample from the Weibull distribution with probability density function $\frac{a}{\theta} x^{a-1} e^{-x^a/\theta} I_{(0,\infty)}(x)$, where $a > 0$ and $\theta > 0$ are unknown. Show that $R(X, a, \theta) = \Pi_{i=1}^n (X_i^a/\theta)$ is pivotal. Construct a confidence set for (a, θ) with confidence coefficient $1 - \alpha$ by using $R(X, a, \theta)$.

~全卷完~

國立中山大學101學年度博士班統計組招生考試試題

科目：【機率論】

注意: 每一題必須將過程清楚寫下來 每一題20分

1. Let $X_1, X_2, \ldots$ be independently identically distributed with mean μ (i.e. $EX_1 = \mu$) and variance σ^2 (i.e. $Var(X_1) = \sigma^2$). Please Prove

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}}} \sim N(0,1),$$

where $\bar{X}$ is the sample mean of $X_1, \ldots, X_n$.

2. Let $X_1, X_2, \ldots,$ be a sequence of random variables and X be a random variable. Please answer the following questions:
(a) If $X_n \to X$ almost surely, prove $X_n \to X$ in probability.
(b) Give an example that $X_n \to X$ in probability but X_n does not converge to X almost surely.

3. Let $X_1, X_2, \ldots$ be independently identically distributed with density $f(x) = 2x^{-3} I_{(1,\infty)}(x)$, where I is an indicator function. Prove $\frac{X_n}{n} \to 0$ almost surely. (Hint: You can use Borel-Cantelli Lemma.)

4. Suppose that X_1 and X_2 are independent gamma variables, and the density functions of X_1 and X_2 are $\frac{1}{\Gamma(\alpha)} x_1^{\alpha-1} \exp(-x_1), 0 < x_1 < \infty$ and $\frac{1}{\Gamma(\beta)} x_2^{\beta-1} \exp(-x_2), 0 < x_2 < \infty$, respectively. Please prove that the distribution of $\frac{X_1}{X_1 + X_2}$ is beta distribution (You need to express the parameters of the beta distribution using α and β).

5. X is a non-negative random variable (i.e. $X \geq 0$) with cumulative distribution function $F(t)$ and expectation $E(X) < \infty$. Please Prove

$$E(X) = \int_0^{\infty} (1 - F(t)) dt.$$

Department of Applied Mathematics, National Sun Yat-sen University

Ph.D. Entrance Examination – Analysis (May, 2012)

Answer all the problems below. The total is 100%. The first problem carries 10%. Each of the rest carries 15%. Unless otherwise stated, f is a real-valued function defined on $\mathbf{R}$.

1. Show that if f and g are continuous real-valued functions on an open set $O \subset \mathbf{R}^n$, then the function h defined by $h(x) = \min\{f(x), g(x)\}$ is also continuous on O.

2. Let g_i $(1 \le i \le n)$ be square integrable functions defined on $[a, b]$. Show that
$$\left[\sum_{i=1}^{n}\left(\int_a^b g_i\right)^2\right]^{1/2} \le \int_a^b \left[\sum_{i=1}^{n} g_i^2\right]^{1/2},$$
and equality holds if and only if all g_i's are multiples of some fixed function ϕ.

3. Let m^* be the Lebesgue outer measure on $\mathbf{R}$. Define
$$m_*(A) = \sup\{m^*(K) : K \text{ is a compact subset of } A\}.$$
Show that if A is measurable and $m(A) < \infty$, then $m_*(A) = m^*(A)$.

4. Give the definition of f being a measurable function. Suppose that for any $c \in \mathbf{R}$, $f^{-1}[\{c\}]$ is a measurable set. Is f a measurable function? Prove or disprove.

5. Let $\{g_n\}$ be a sequence of nonnegative integrable functions defined on a measurable set $E \subset \mathbf{R}$, which converges a.e. to an integrable function g. Let $\{f_n\}$ be a sequence of measurable functions on E such that $|f_n| \le g_n$ and f_n converges to f a.e. $x \in E$. Show that if $\int_E g = \lim_{n\to\infty} \int_E g_n$, then
$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

6. Let E be a measurable set and $1 < p < \infty$. Suppose that $\{f_n\}$ converges to f weakly. Show that there is a subsequence $\{f_{n_k}\}$ such that

$$\lim_{k\to\infty} \frac{f_{n_1} + f_{n_2} + \cdots + f_{n_k}}{k} = f \quad \text{strongly in } L^p(E).$$

7. Show that any compact subset of a metric space has to be closed and bounded.

End of Paper

組合數學(Combinatorics)

May 2012

注意： 每個問題必需證明或說明清楚。

1. Let a_n be the way to distribute n identical balls into five distinct boxes with the first box having at most two balls.
   (a) Find the generating function for a_n. (10%)
   (b) Find a_{505}. (10%)

2. Let S={1,2,3,4,5}, A be the set of all permutations of S, and B={(a,b,c,d,e)∈A : $a \notin \{1,5\}$, $b \notin \{2,3\}$, $c \notin \{3,4\}$, $d \notin \{4,5\}$, and $e \notin \{3,5\}$}. Find the cardinality of the set B. (20%)

3. Let G be a simple planar graph. Prove that if every subgraph of G has a vertex of degree ≤ 5, then G is 5-colorable. (20%)

4. Prove Hall's Theorem by two different methods. (20%)

5. Let n be a positive integer and $[n]$ be the set $\{1,2,\ldots,n\}$. Define G as a graph with vertex set of G being all subsets of $[n]$, and edge set {AB: A is a subset of B}. Find a maximum independent set of G. (20%)